

COCOMPLETION OF RESTRICTION CATEGORIES

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ABSTRACT. Restriction categories were introduced as a way of generalising the notion of partial map categories. In this paper, we define cocomplete restriction category, and give the free cocompletion of a small restriction category as a suitably defined category of restriction presheaves. We also consider the case where our restriction category is locally small.

1. INTRODUCTION

The notion of a partial function is ubiquitous in many areas of mathematics, most notably in computability theory, complexity theory, algebraic geometry and algebraic topology. However, such notions of partiality need not be solely restricted to sets and partial functions between them, but may also arise in the context of continuous functions on the open subsets of topological spaces [11, p. 97]. An early attempt at describing an abstract notion of partiality came from Carboni [1], who considered bicategories with a tensor product and a unique cocommutative comonoid structure. However, the first real attempt at axiomatising the general theory came from Di Paola and Heller [7], who introduced the notion of a *dominical category*. Around the same time, Robinson and Rosolini [11] gave their own interpretation of this notion of partiality through what they called *p-categories*, and observed that Di Paola and Heller’s dominical categories were in fact instances of *p-categories*.

The common theme between dominical categories and *p-categories* is their reliance on classes of monomorphisms for partiality. However, it was shown by Grandis [8] that it was possible to capture the partiality of maps in the form of idempotents on their domains, via the notion of *e-cohesive categories*. This same idea was later reformulated and studied extensively by Cockett and Lack in their series of three papers on *restriction categories* [3, 4, 5]. Informally, in a restriction category \mathbf{X} , the restriction of a map $f: A \rightarrow B \in \mathbf{X}$ is an idempotent $\bar{f}: A \rightarrow A$ which measures the degree of the partiality of f . In particular, in the category of sets and partial functions, the restriction of a map $f: A \rightarrow B$ is a partial identity map on A which has the same domain of definition as f .

Since restriction categories are categories with extra structure, it would not be too far-fetched to think that one could give a notion of colimits in this restriction setting. As a matter of fact, Cockett and Lack give an explicit description of

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restriction coproducts in a restriction category [5, Lemma 2.1]. As a necessary first step towards understanding restriction colimits in general, we consider the notion of a *cocomplete restriction category*, and of free restriction cocompletion; indeed this is what we will do in this paper. Future work will include extending this notion of restriction cocompletion to join restriction categories, and showing that the manifold completion of a join restriction category [8] is a full subcategory of this join restriction cocompletion, whatever that might be. Another possibility is to extend this to categories with a restriction tangent structure, and showing that its free cocompletion also has a restriction tangent structure [2].

The starting point for our discussion will be a revision of background material from Cockett and Lack [3], in section 2. In section 3, we define cocomplete \mathcal{M} -category and cocontinuous \mathcal{M} -functors. Then using the fact \mathcal{M} -categories are *the same* as split restriction categories, we give a definition of cocomplete restriction category and cocontinuous restriction functors. We also show that the Cockett-Lack embedding exhibits the split restriction category $\text{Par}(\text{PSh}_{\mathcal{M}}(\mathcal{M}\text{Total}(\mathbf{K}_r(\mathbf{X}))))$ as the free cocompletion of any small restriction category \mathbf{X} .

In section 4, we introduce the notion of restriction presheaf on a restriction category \mathbf{X} , and give an explicit description of the split restriction category of restriction presheaves $\text{PSh}_r(\mathbf{X})$. Finally, we show that this restriction presheaf category $\text{PSh}_r(\mathbf{X})$ is in fact equivalent to $\text{Par}(\text{PSh}_{\mathcal{M}}(\mathcal{M}\text{Total}(\mathbf{K}_r(\mathbf{X}))))$, and this in turn gives us an alternate formulation of restriction free cocompletion.

Finally in section 5, we consider the case where our \mathcal{M} -category \mathbf{C} may not be small, but locally small, and give a definition of what it means for an \mathcal{M} -category to be locally small. We see that for any locally small \mathcal{M} -category \mathbf{C} , the \mathcal{M} -category of small presheaves $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is not only locally small and cocomplete, but is also the free cocompletion of \mathbf{C} . Then as before, it turns out that for any locally small restriction category \mathbf{X} , the Cockett-Lack embedding exhibits the restriction category $\text{Par}(\mathcal{P}_{\mathcal{M}}(\mathcal{M}\text{Total}(\mathbf{K}_r(\mathbf{X}))))$ as its free cocompletion. Also, just as small presheaves are defined to be a small colimit of representables, we define small restriction presheaves analogously.

2. RESTRICTION CATEGORY PRELIMINARIES

2.1. Restriction categories. In this section, we recall the definition of a restriction category and basic lemmas from [3]. We recall there is a 2-category of restriction categories called \mathbf{rCat} , and that \mathbf{rCat} has an important sub-2-category \mathbf{rCat}_s of split restriction categories. The reason for its importance is due to [3, Theorem 3.4], which says there is an equivalence between \mathbf{rCat}_s and the 2-category \mathcal{MCat} of \mathcal{M} -categories (or categories with a stable system of monics). A consequence of this theorem is that it allows us to work with \mathcal{M} -categories, which are not much different to ordinary categories, and transfer any results obtained across to restriction categories. We will be referring frequently to this equivalence between \mathbf{rCat}_s and \mathcal{MCat} in later sections.

Definition 1. A *restriction category* is a category \mathbf{X} together with assignments

$$\mathbf{X}(A, B) \rightarrow \mathbf{X}(A, A), \quad f \mapsto \bar{f}$$

where \bar{f} satisfies the following conditions:

- (R1) $f \circ \bar{f} = f$;
- (R2) $\bar{g} \circ \bar{f} = \bar{f} \circ \bar{g}$ for $f: A \rightarrow B, g: A \rightarrow C$;
- (R3) $\overline{g \circ f} = \bar{g} \circ \bar{f}$ for $f: A \rightarrow B, g: A \rightarrow C$;
- (R4) $\bar{h} \circ f = f \circ \bar{h} \circ \bar{f}$ for $f: A \rightarrow B, h: B \rightarrow C$.

The assignments $f \mapsto \bar{f}$ are called the *restriction structure* on \mathbf{X} , and we call \bar{f} the *restriction* of f .

Examples 2. (1) The category of sets and partial functions \mathbf{Set}_p is a restriction category, where the restriction on each partial function $f: A \rightarrow B$ is given by

$$\bar{f}(a) = \begin{cases} a & \text{if } f \text{ is defined at } a \in A; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

- (2) Consider the set of natural numbers \mathbb{N} as a monoid whose composition is given by $n \circ m = \max(m, n)$. Then \mathbb{N} may be given two restriction structures; the first by $\bar{n} = n$, and the second by

$$\bar{n} = \begin{cases} n & n = 0 \text{ or } n \text{ odd}; \\ n - 1 & \text{otherwise.} \end{cases}$$

The restriction \bar{f} of any map f in a restriction category satisfies the following basic properties (see [3, pp. 227,230] for details).

Lemma 3. *Let \mathbf{X} be a restriction category, and let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms in \mathbf{X} . Then*

- (1) \bar{f} is idempotent;
- (2) $\bar{f} \circ \overline{gf} = \overline{gf}$;
- (3) $\overline{gf} = \overline{gf}$;
- (4) $\bar{f} = \overline{f}$;
- (5) if f is a monomorphism, then $\bar{f} = 1$;
- (6) $\mathbf{X}(A, B)$ has a partial order given by $f \leq f'$ if and only if $f = f' \circ \bar{f}$.

A map $f \in \mathbf{X}$ is called a *restriction idempotent* if $f = \bar{f}$, and is *total* if $\bar{f} = 1$. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are total maps in a restriction category, then gf is also total since

$$\overline{gf} = \overline{gf} = \bar{f} = 1.$$

Therefore, as identities are total, the objects and total maps of any restriction category \mathbf{X} form a subcategory $\mathbf{Total}(\mathbf{X})$.

Definition 4. A functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ between restriction categories is called a *restriction functor* if $F(\bar{f}) = \overline{F(f)}$ for all maps $f \in \mathbf{X}$, and a natural transformation $\alpha: F \Rightarrow G$ is a *restriction transformation* if its components are total. We denote by \mathbf{rCat} the 2-category of restriction categories (objects), restriction functors (1-cells) and restriction transformations (2-cells).

2.2. Split restriction categories. There is an important full sub-2-category \mathbf{rCat}_s of \mathbf{rCat} , the objects of which are restriction categories whose restriction idempotents split. Recall that a restriction idempotent \bar{f} *splits* if there exist maps m and r such that $mr = \bar{f}$ and $rm = 1$. We call such maps m , *restriction monics*.

The inclusion $\mathbf{rCat}_s \hookrightarrow \mathbf{rCat}$ has a left biadjoint K_r , which on objects takes restriction categories \mathbf{X} to split restriction categories $K_r(\mathbf{X})$ [3, p. 242] with the following data:

Objects: Pairs (A, e) , where A is an object of \mathbf{X} and $e: A \rightarrow A$ is a restriction idempotent on A ;

Morphisms: Morphisms $f: (A, e) \rightarrow (A', e')$ are morphisms $f: A \rightarrow A'$ in \mathbf{X} satisfying the condition $e'fe = f$;

Restriction: Restriction on f is given by \bar{f} .

The unit at \mathbf{X} of this biadjunction, $J: \mathbf{X} \rightarrow K_r(\mathbf{X})$, takes an object A to $(A, 1_A)$ and a morphism $f: A \rightarrow A'$ to $f: (A, 1_A) \rightarrow (A', 1_{A'})$ in $K_r(\mathbf{X})$. As alluded to earlier, this 2-category of split restriction categories \mathbf{rCat}_s is equivalent to a 2-category called \mathcal{MCat} , the objects of which form the basis for our discussion in the next section.

2.3. \mathcal{M} -categories and partial map categories. A *stable system of monics* $\mathcal{M}_{\mathbf{C}}$ in a category \mathbf{C} is a collection of monics in \mathbf{C} which includes all isomorphisms, is closed under composition, and the pullback of any $m \in \mathcal{M}_{\mathbf{C}}$ along arbitrary maps in \mathbf{C} exists and is in $\mathcal{M}_{\mathbf{C}}$. An \mathcal{M} -category is then a category \mathbf{C} together with a stable system of monics $\mathcal{M}_{\mathbf{C}} \subseteq \mathbf{C}$, which we write as a pair $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ [3, p. 245]. (Where the meaning is clear, we shall dispense with the notation $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ and simply write \mathbf{C}).

If \mathbf{C} and \mathbf{D} are \mathcal{M} -categories, a functor F between them is called an \mathcal{M} -functor if $m \in \mathcal{M}_{\mathbf{C}}$ implies $Fm \in \mathcal{M}_{\mathbf{D}}$, and F preserves pullbacks of monics in $\mathcal{M}_{\mathbf{C}}$. Further, if $F, G: \mathbf{C} \rightarrow \mathbf{D}$ are \mathcal{M} -functors, a natural transformation between them is called \mathcal{M} -cartesian if the naturality square is a pullback for all $m \in \mathcal{M}_{\mathbf{C}}$ [3, p. 247]. We denote by \mathcal{MCat} the 2-category of \mathcal{M} -categories (objects), \mathcal{M} -functors (1-cells) and \mathcal{M} -cartesian natural transformations (2-cells).

Now associated with any \mathcal{M} -category \mathbf{C} is the split restriction category $\mathbf{Par}(\mathbf{C})$, called the *category of partial maps in \mathbf{C}* . It has the same objects as \mathbf{C} , and morphisms from $X \rightarrow Y$ in $\mathbf{Par}(\mathbf{C})$ are spans

$$(m, f) = X \xleftarrow{m \in \mathcal{M}_{\mathbf{C}}} Z \xrightarrow{f} Y$$

identified up to some equivalence class. More precisely, $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ such that $m\varphi = n$ and $f\varphi = g$. Composition in this category is by pullback, the identity is given by $(1, 1)$ and the restriction of (m, f) is (m, m) [3, pp. 246, 247].

There is also a 2-functor $\mathbf{Par}: \mathcal{MCat} \rightarrow \mathbf{rCat}_s$ which on objects, takes \mathcal{M} -categories \mathbf{C} to split restriction categories $\mathbf{Par}(\mathbf{C})$. If $F: \mathbf{C} \rightarrow \mathbf{D}$ is an \mathcal{M} -functor, then $\mathbf{Par}(F)$ takes objects $A \in \mathbf{Par}(\mathbf{C})$ to FA and morphisms (m, f) to (Fm, Ff) . Also, if $\alpha: F \Rightarrow G$ is \mathcal{M} -cartesian, then $\mathbf{Par}(\alpha)$ is defined componentwise by $\mathbf{Par}(\alpha)_A = (1_{FA}, \alpha_A)$.

Theorem 5. *The 2-functor $\mathbf{Par}: \mathcal{MCat} \rightarrow \mathbf{rCat}_s$ is an equivalence of 2-categories.*

Proof. We give a quick sketch of the proof. For full details, see [3, Theorem 3.4]. We know that Par is a 2-functor from $\mathcal{M}\text{Cat}$ to \mathbf{rCat}_s . Likewise, there is a 2-functor $\mathcal{M}\text{Total}: \mathbf{rCat}_s \rightarrow \mathcal{M}\text{Cat}$, taking split restriction categories \mathbf{X} to \mathcal{M} -categories $(\text{Total}(\mathbf{X}), \mathcal{M}_{\text{Total}(\mathbf{X})})$, where $\mathcal{M}_{\text{Total}(\mathbf{X})}$ consists of the restriction monics in \mathbf{X} . (Recall that $\mathcal{M}\text{Total}(\mathbf{X})$ really is an \mathcal{M} -category [3, Proposition 3.3]).

The pair Par and $\mathcal{M}\text{Total}$ are then part of a 2-equivalence, with the unit at \mathbf{X} , $\Phi_{\mathbf{X}}: \mathbf{X} \rightarrow \text{Par}(\mathcal{M}\text{Total}(\mathbf{X}))$, given by $\Phi_{\mathbf{X}}(A) = A$ on objects and by $\Phi_{\mathbf{X}}(f) = (m, fm)$ on arrows (where $\bar{f} = mr$ and $rm = 1$). On the other hand, the counit at \mathbf{C} is defined by $\Psi_{\mathbf{C}}(A) = A$ on objects and $\Psi_{\mathbf{C}}(1, f) = f$ on morphisms. \square

3. COCOMPLETION OF RESTRICTION CATEGORIES

For any small category \mathbf{C} , we may characterise the category of presheaves $\text{PSh}(\mathbf{C})$ as the *free cocompletion* of \mathbf{C} . That is, for any small-cocomplete category \mathcal{E} , the following is an equivalence of categories:

$$(-) \circ \mathbf{y}: \mathbf{Cocomp}(\text{PSh}(\mathbf{C}), \mathcal{E}) \rightarrow \mathbf{Cat}(\mathbf{C}, \mathcal{E})$$

where \mathbf{y} is the Yoneda embedding, \mathbf{Cat} is the 2-category of small categories and \mathbf{Cocomp} is the 2-category of small-cocomplete categories and cocontinuous functors. (For the rest of this paper, we shall take *cocomplete* to mean *small-cocomplete*, and *colimits* to mean *small colimits* unless otherwise indicated). However, it is not immediately obvious that there is an analogous notion of cocompletion for any small restriction category \mathbf{X} . Nonetheless, a clue is given to us in light of the 2-equivalence between $\mathcal{M}\text{Cat}$ and \mathbf{rCat}_s . That is, it might be helpful to first define a notion of cocomplete \mathcal{M} -category, and study the free cocompletion of small \mathcal{M} -categories.

In this section, we recall the \mathcal{M} -category of presheaves $\text{PSh}_{\mathcal{M}}(\mathbf{C})$ for any small \mathcal{M} -category \mathbf{C} and give a definition of cocomplete \mathcal{M} -category and cocontinuous \mathcal{M} -functor. (As it turns out, this \mathcal{M} -category of presheaves, $\text{PSh}_{\mathcal{M}}(\mathbf{C})$ will be the free cocompletion of any small \mathcal{M} -category \mathbf{C}). Then using the 2-equivalence between $\mathcal{M}\text{Cat}$ and \mathbf{rCat}_s , we define cocomplete restriction categories and cocontinuous restriction functors. This in turn provides a candidate for free restriction cocompletion, namely the split restriction category $\text{Par}(\text{PSh}_{\mathcal{M}}(\mathcal{M}\text{Total}(\mathbf{K}_r(\mathbf{X}))))$ described by Cockett and Lack [3].

3.1. An \mathcal{M} -category of presheaves. For any small \mathcal{M} -category \mathbf{C} , there are various ways of constructing an \mathcal{M} -category of presheaves on \mathbf{C} . One way is the following, and we denote the \mathcal{M} -category arising in this way by $\text{PSh}_{\mathcal{M}}(\mathbf{C}) = (\text{PSh}(\mathbf{C}), \mathcal{M}_{\text{PSh}(\mathbf{C})})$. We say a map $\mu: P \Rightarrow Q$ is an $\mathcal{M}_{\text{PSh}(\mathbf{C})}$ -map if for all $\gamma: \mathbf{y}D \Rightarrow Q$, there exists an $m \in \mathcal{M}_{\mathbf{C}}$ making the following a pullback square:

$$\begin{array}{ccc} \mathbf{y}C & \longrightarrow & P \\ \mathbf{y}m \downarrow & & \downarrow \mu \\ \mathbf{y}D & \xrightarrow{\gamma} & Q \end{array}$$

where $\mathbf{y}: \mathbf{C} \rightarrow \mathbf{PSh}(\mathbf{C})$ is the usual Yoneda embedding [3, p. 252]. Observe that under this construction, the Yoneda embedding is an \mathcal{M} -functor $\mathbf{y}: \mathbf{C} \rightarrow \mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$.

3.2. Cocomplete \mathcal{M} -categories. It is well known that for any small \mathcal{M} -category \mathbf{C} , the Yoneda embedding $\mathbf{y}: \mathbf{C} \rightarrow \mathbf{PSh}(\mathbf{C})$ exhibits $\mathbf{PSh}(\mathbf{C})$ as the free cocompletion of \mathbf{C} . Therefore it is natural to ask whether for any small \mathcal{M} -category \mathbf{C} , the Yoneda embedding $\mathbf{y}: \mathbf{C} \rightarrow \mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ likewise exhibits $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ as the free cocompletion of \mathbf{C} . First we need to give a definition of cocomplete \mathcal{M} -category and cocontinuous \mathcal{M} -functor.

Definition 6. An \mathcal{M} -category $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ is cocomplete if \mathbf{C} is itself cocomplete and its inclusion into $\mathbf{Par}(\mathbf{C})$ preserves colimits. An \mathcal{M} -functor $F: (\mathbf{C}, \mathcal{M}_{\mathbf{C}}) \rightarrow (\mathbf{D}, \mathcal{M}_{\mathbf{D}})$ between \mathcal{M} -categories is cocontinuous if the underlying functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is cocontinuous. We denote by $\mathcal{MCocomp}$ the 2-category of cocomplete \mathcal{M} -categories, cocontinuous \mathcal{M} -functors and \mathcal{M} -cartesian natural transformations.

Example 7. Let \mathbf{Set} denote the category of all small sets, and consider the \mathcal{M} -category $(\mathbf{Set}, \mathcal{M}_{\mathbf{Set}})$, where $\mathcal{M}_{\mathbf{Set}}$ are all the injective functions. Then, as \mathbf{Set} is cocomplete and $\mathbf{Set} \hookrightarrow \mathbf{Par}(\mathbf{Set}, \mathcal{M}_{\mathbf{Set}}) = \mathbf{Set}_p$ has a right adjoint, $(\mathbf{Set}, \mathcal{M}_{\mathbf{Set}})$ is a cocomplete \mathcal{M} -category.

As a matter of fact, there are whole classes of examples of cocomplete \mathcal{M} -categories. Before we give their construction, it will be helpful to define what we mean by an \mathcal{M} -subobject.

Definition 8. (\mathcal{M} -subobjects) Let \mathbf{C} be an \mathcal{M} -category and D an object in \mathbf{C} . Then an \mathcal{M} -subobject is an isomorphism class of $\mathcal{M}_{\mathbf{C}}$ -maps with codomain D . That is, if $m: C \rightarrow D$ and $m': C' \rightarrow D$ are both in $\mathcal{M}_{\mathbf{C}}$, then m and m' represent the same subobject of D if there exists an isomorphism $\varphi: C \rightarrow C'$ such that $m = m'\varphi$. We shall use the notation $\mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(D)$ to denote the set of subobjects of D in the \mathcal{M} -category \mathbf{C} .

It will be useful to observe the following lemma in relation to \mathcal{M} -subobjects of representables in the \mathcal{M} -category $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$.

Lemma 9. *Let \mathbf{C} be an \mathcal{M} -category. Then there exists an isomorphism as follows:*

$$\mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C) \cong \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(C).$$

Proof. Define a function $\varphi: \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(C) \rightarrow \mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C)$, which takes an \mathcal{M} -subobject $m: D \rightarrow C$ to $\mathbf{y}m: \mathbf{y}D \rightarrow \mathbf{y}C$, a map in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$. To define its inverse, consider the function $\psi: \mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C) \rightarrow \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(C)$ which takes an \mathcal{M} -subobject of $\mathbf{y}C$, $\mu: P \rightarrow \mathbf{y}C$, to the unique subobject $n: A \rightarrow C$ making the diagram on the left a pullback:

$$\begin{array}{ccc}
\mathbf{y}A & \xrightarrow{\alpha} & P \\
\mathbf{y}n \downarrow & & \downarrow \mu \\
\mathbf{y}C & \xrightarrow{1_{\mathbf{y}C}} & \mathbf{y}C
\end{array}
\qquad
\begin{array}{ccc}
& & 1 \\
& \searrow \beta & \nearrow \\
P & & P \\
\mu \swarrow & \mathbf{y}A \xrightarrow{\alpha} & \searrow \mu \\
& \mathbf{y}n \downarrow & \\
& \mathbf{y}C \xrightarrow{1_{\mathbf{y}C}} & \mathbf{y}C
\end{array}$$

Clearly $\psi \circ \varphi = 1$. To see that $\varphi \circ \psi = 1$, consider the previous diagram on the right. By definition, there exists a unique map β such that $\mathbf{y}n \circ \beta = \mu$ and $\alpha\beta = 1_P$. But $\mathbf{y}n = \mathbf{y}n \circ \beta \circ \alpha$ and $\mathbf{y}n$ is monic, which means $\beta\alpha = 1$. Therefore, μ and $\mathbf{y}n$ belong to the same isomorphism class of \mathcal{M} -subobjects of $\mathbf{y}C$. Hence $\varphi \circ \psi = 1$, and so $\mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C) \cong \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(C)$. \square

Now consider an \mathcal{M} -category $(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$, where $\mathcal{M}_{\mathcal{E}}$ is a stable system of monics and \mathcal{E} is a cocomplete category with a terminal object 1 and a generic \mathcal{M} -subobject $\tau: 1 \rightarrow \Sigma$. By a generic \mathcal{M} -subobject (or an \mathcal{M} -subobject classifier), we mean an object $\Sigma \in \mathcal{E}$ and an $\mathcal{M}_{\mathcal{E}}$ -map $\tau: 1 \rightarrow \Sigma$ such that for any $\mathcal{M}_{\mathcal{E}}$ -map $m: A \rightarrow B$, there exists a unique map $\tilde{m}: B \rightarrow \Sigma$ making the following square a pullback:

$$\begin{array}{ccc}
A & \longrightarrow & 1 \\
m \downarrow & & \downarrow \tau \\
B & \xrightarrow{\tilde{m}} & \Sigma
\end{array}$$

Suppose the induced pullback functor $\tau^*: \mathcal{E}/\Sigma \rightarrow \mathcal{E}$ has a right adjoint Π_{τ} . Then by an analogous argument in topos theory [9, Proposition 2.4.7], \mathcal{E} has a *partial map classifier* for every object $C \in \mathcal{E}$, and this in turn implies that the inclusion $\mathcal{E} \hookrightarrow \mathbf{Par}(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$ has a right adjoint [4, p. 65], and so \mathcal{M} -categories of this kind are cocomplete.

- Examples 10.** (1) Let \mathcal{E} be any cocomplete elementary topos, and let $\mathcal{M}_{\mathcal{E}}$ be all the monics in \mathcal{E} . Then $(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$ is a cocomplete \mathcal{M} -category since \mathcal{E} is locally cartesian closed and has a generic subobject.
- (2) If \mathcal{E} is any cocomplete quasitopos and $\mathcal{M}_{\mathcal{E}}$ are all the regular monics in \mathcal{E} , then $(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$ is also a cocomplete \mathcal{M} -category as it is locally cartesian closed and has an object which classifies all the regular monics in \mathcal{E} .
- (3) We know the presheaf category on any small category \mathbf{C} is cocomplete and locally cartesian closed. So consider the \mathcal{M} -category $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$. If an \mathcal{M} -subobject classifier were to exist, then by Yoneda, we would have

$$\Sigma(C) \cong \mathbf{PSh}(\mathbf{C})(\mathbf{y}C, \Sigma) \cong \mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C).$$

But because $\mathbf{Sub}_{\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}}(\mathbf{y}C) \cong \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}(C)$ (Lemma 9), define Σ to take objects $C \in \mathbf{C}$ to the set of \mathcal{M} -subobjects of C , and maps $f: D \rightarrow C$ in \mathbf{C} to f^* , the change-of-base functor (by pullback along f). Finally, define the map $\tau: 1 \rightarrow \Sigma$ componentwise at $C \in \mathbf{C}$ by taking the singleton to the largest \mathcal{M} -subobject of C , the identity on C .

It is then not difficult to check that this map $\tau: 1 \rightarrow \Sigma$ is in $\mathcal{M}_{\text{PSh}(\mathbf{C})}$, and also classifies all $\mathcal{M}_{\text{PSh}(\mathbf{C})}$ -maps. Hence, $\text{PSh}_{\mathcal{M}}(\mathbf{C})$ is a cocomplete \mathcal{M} -category.

The following proposition gives an alternative characterisation of the inclusion $\mathbf{C} \hookrightarrow \text{Par}(\mathbf{C})$ being cocontinuous for a cocomplete category \mathbf{C} .

Proposition 11. *Suppose $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ is an \mathcal{M} -category, and \mathbf{C} is cocomplete. Then the following statements are equivalent:*

- (1) *The inclusion $\mathbf{C} \hookrightarrow \text{Par}(\mathbf{C})$ preserves colimits;*
- (2) *The following conditions hold:*
 - (a) *If $\{m_i: A_i \rightarrow B_i\}_{i \in I}$ is a family of maps in $\mathcal{M}_{\mathbf{C}}$ indexed by a small set I , then their coproduct $\sum_{i \in I} m_i$ is in $\mathcal{M}_{\mathbf{C}}$ and the following squares are pullbacks for every $i \in I$:*

$$\begin{array}{ccc} A_i & \xrightarrow{v_{A_i}} & \sum_{i \in I} A_i \\ m_i \downarrow & & \downarrow \sum_{i \in I} m_i \\ B_i & \xrightarrow{v_{B_i}} & \sum_{i \in I} B_i \end{array}$$

- (b) *Suppose $m \in \mathcal{M}_{\mathbf{C}}$ and the pullback of m along two maps $f, g \in \mathbf{C}$ is the same map h . If f', g' are the pullbacks of f, g along m , and c, c' are the coequalisers of f, g and f', g' respectively, then the unique n making the right square commute is in $\mathcal{M}_{\mathbf{C}}$ and also makes the right square a pullback:*

$$\begin{array}{ccccc} \bullet & \xrightarrow{f'} & \bullet & \xrightarrow{c'} & \bullet \\ \downarrow h & \Downarrow g' & \downarrow m & & \downarrow n \\ \bullet & \xrightarrow{f} & \bullet & \xrightarrow{c} & \bullet \\ & \Downarrow g & & & \end{array}$$

- (c) *Colimits are stable under pullback along $\mathcal{M}_{\mathbf{C}}$ -maps.*

Proof. For the proof of (1) \implies (2), we will be using Lemma 16 and Corollary 18 (both to be proven later).

(1) \implies (2a) Let I be a small discrete category, and let $H, K: I \rightarrow \mathbf{C}$ be functors taking objects $i \in I$ to A_i and B_i respectively. Let $\alpha: H \Rightarrow K$ be a natural transformation whose component at i is given by $m_i: A_i \rightarrow B_i$, and observe that all naturality squares are trivially pullbacks. Then by Lemma 16, the sum $\sum_{i \in I} m_i$ is in $\mathcal{M}_{\mathbf{C}}$ and for every $i \in I$, the coproduct coprojection squares are pullbacks.

(1) \implies (2b) Take I to be the category with two objects and a pair of parallel maps between them and apply Lemma 16.

(1) \implies (2c) See Corollary 18.

(2) \implies (1) Conversely, to show that the inclusion $\mathbf{C} \hookrightarrow \text{Par}(\mathbf{C})$ is cocontinuous, it is enough to show that it preserves all small coproducts and coequalisers.

So suppose c is a coequaliser of f and g in \mathbf{C} .

$$\bullet \xrightarrow[\Downarrow g]{f} \bullet \xrightarrow{c} \bullet$$

To show the inclusion preserves this coequaliser, we need to show that for any map (m, k) such that $(m, k)(1, f) = (m, k)(1, g)$, there is a unique map (n, q) making the following diagram commute:

$$\begin{array}{ccccc}
 \bullet & \xrightarrow[(1, g)]{(1, f)} & \bullet & \xrightarrow{(1, c)} & \bullet \\
 & & & \searrow (m, k) & \downarrow (n, q) \\
 & & & & \bullet
 \end{array}$$

Now the condition $(m, k)(1, f) = (m, k)(1, g)$ is precisely the condition that the pullbacks of m along f and g are the same map h ,

$$\begin{array}{ccc}
 \bullet & \xrightarrow[f']{f'} & \bullet \\
 h \downarrow & \xrightarrow[g']{g'} & \downarrow m \\
 \bullet & \xrightarrow[g]{f} & \bullet
 \end{array}$$

and that $kf' = kg'$. Taking c' to be the coequaliser of f' and g' , our assumption then implies there is a unique map $n \in \mathcal{M}_{\mathbf{C}}$ making the following diagram a pullback:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{c'} & \bullet \\
 m \downarrow & & \downarrow n \\
 \bullet & \xrightarrow{c} & \bullet
 \end{array}$$

Since c' is the coequaliser of f' and g' and $kf' = kg'$, there exists a unique map q such that $c'q = k$. This gives a map $(n, q) \in \mathbf{Par}(\mathbf{C})$ such that $(n, q)(1, c) = (m, k)$. To see it must be unique, suppose (n', q') also satisfies the condition $(n', q')(1, c) = (m, k)$. By assumption, as colimits are stable under pullback along $\mathcal{M}_{\mathbf{C}}$ -maps, the pullback of c along n' must be a coequaliser of f' and g' , say c'' .

$$\begin{array}{ccc}
 \bullet & \xrightarrow{c''} & \bullet \\
 m \downarrow & \searrow c' \quad \nearrow \varphi & \downarrow n' \\
 \bullet & & \bullet \\
 & \searrow n & \\
 \bullet & \xrightarrow{c} & \bullet
 \end{array}$$

Now as coequalisers are unique up to isomorphism, there is an isomorphism φ such that $c'' = \varphi c'$. But the fact

$$n'\varphi c' = n'c'' = cm = nc'$$

implies $n'\varphi = n$ as c' is an epimorphism. In other words, n and n' must be the same \mathcal{M} -subobject. Similarly, $q = q'\varphi$, which means $(n, q) = (n', q')$.

Next, suppose $\sum_{i \in I} B_i$ is a small coproduct in \mathbf{C} , with coproduct coprojections $(\iota_{B_i} : B_i \rightarrow \sum_{i \in I} B_i)_{i \in I}$. Then $\sum_{i \in I} B_i$ will be a small coproduct in $\mathbf{Par}(\mathbf{C})$ if for

any object $D \in \mathbf{Par}(\mathbf{C})$ and family of maps $((m_i, f_i): B_i \rightarrow D)_{i \in I}$, there exists a unique map $(\mu, \gamma): \sum_{i \in I} B_i \rightarrow D$ making the following diagram commute for every $i \in I$:

$$\begin{array}{ccc} B_i & \xrightarrow{(1, \iota_{B_i})} & \sum_{i \in I} B_i \\ & \searrow (m_i, f_i) & \downarrow (\mu, \gamma) \\ & & D \end{array}$$

By assumption, $\sum_{i \in I} m_i$ is in $\mathcal{M}_{\mathbf{C}}$, and so the map $(\sum_{i \in I} m_i, f): \sum_{i \in I} B_i \rightarrow D$ is well-defined, where f is the unique map $\sum_{i \in I} \mathbf{dom}(f_i) \rightarrow D$ induced by the universal property of the coproduct coprojections and the family of maps $\{f_i\}_{i \in I}$. Since the coproduct coprojection squares are pullbacks, taking $\mu = \sum_{i \in I} m_i$ and $\gamma = f$ certainly makes the above diagram commute, and the uniqueness of (μ, γ) follows by an analogous argument to the case of coequalisers by the stability of colimits under pullback. Therefore, if $\sum_{i \in I} B_i$ is a small coproduct in \mathbf{C} , it is also a small coproduct in $\mathbf{Par}(\mathbf{C})$.

Therefore, since the inclusion $\mathbf{C} \hookrightarrow \mathbf{Par}(\mathbf{C})$ preserves all small coproducts and all coequalisers, it preserves all small colimits. \square

Remark 12. There is yet another formulation for the condition that the inclusion $\mathbf{C} \hookrightarrow \mathbf{Par}(\mathbf{C})$ preserves all small colimits. That is, the inclusion is cocontinuous if and only if the functor $\mathbf{Sub}_{\mathcal{M}_{\mathbf{C}}}: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$, which on objects takes C to the set of \mathcal{M} -subobjects of C , is continuous, and moreover, colimits are stable under pullback along $\mathcal{M}_{\mathbf{C}}$ -maps. The proof of this result is similar to the proof of Lemma 11.

Also, by conditions (2a) and (2c), observe that cocomplete \mathcal{M} -categories must be \mathcal{M} -extensive, meaning that for every $i \in I$ (with I small), if the following square is commutative with the bottom row being coproduct injections and $m, m_i \in \mathcal{M}$ (for all $i \in I$), then the top row must be a coproduct diagram if and only if each square is a pullback:

$$\begin{array}{ccc} A_i & \longrightarrow & Z \\ m_i \downarrow & & \downarrow m \\ B_i & \xrightarrow{\iota_{B_i}} & \sum_{i \in I} B_i \end{array}$$

In light of the previous proposition, we give an example of an \mathcal{M} -category which is not cocomplete.

Example 13. Consider the \mathcal{M} -category $(\mathbf{Ab}, \mathcal{M}_{\mathbf{Ab}})$ of small abelian groups and all monomorphisms in \mathbf{Ab} . Denote the trivial group by 0 and the group of integers by \mathbb{Z} . The coproduct of \mathbb{Z} with itself is just the direct sum $\mathbb{Z} \oplus \mathbb{Z}$, along with coprojections $\iota_1: \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ and $\iota_2: \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ sending n to $(n, 0)$ and $(0, n)$ respectively. Let $\Delta: \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ denote the diagonal map, which is clearly a monomorphism and hence lies in $\mathcal{M}_{\mathbf{Ab}}$. Now a pullback of Δ along ι_1 is the unique map $0 \rightarrow \mathbb{Z}$, and similarly for ι_2 . This gives the following diagram, where both squares are pullbacks:

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z} & \longleftarrow & 0 \\
\downarrow & & \downarrow \Delta & & \downarrow \\
\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{i_2} & \mathbb{Z}
\end{array}$$

However, the top row is certainly not a coproduct diagram in \mathbf{Ab} . Therefore, $(\mathbf{Ab}, \mathcal{M}_{\mathbf{Ab}})$ is not \mathcal{M} -extensive, and hence by Proposition 11, is not a cocomplete \mathcal{M} -category.

3.3. Cocompletion of \mathcal{M} -categories. Our goal is to show for any small \mathcal{M} -category \mathbf{C} and cocomplete \mathcal{M} -category \mathbf{D} , the following is an equivalence:

$$(-) \circ \mathbf{y}: \mathcal{M}\mathbf{Cocomp}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \rightarrow \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D}).$$

To do so will require the next four lemmas.

Lemma 14. *Let \mathbf{C} be an \mathcal{M} -category and let $m \in \mathcal{M}_{\mathbf{C}}$. Then the following is a pullback square*

$$\begin{array}{ccc}
A & \xrightarrow{g} & B \\
n \downarrow & & \downarrow m \\
C & \xrightarrow{f} & D
\end{array}$$

if and only if the following diagram commutes in $\mathbf{Par}(\mathbf{C})$:

$$\begin{array}{ccc}
C & \xrightarrow{(1, f)} & D \\
(n, 1) \downarrow & & \downarrow (m, 1) \\
A & \xrightarrow{(1, g)} & B
\end{array}$$

Proof. Diagram chase. □

Lemma 15. *Let \mathbf{X} be a restriction category and \mathbf{I} any small category. Suppose given $L: \mathbf{I} \rightarrow \mathbf{X}$ and a colimiting cocone $p_I: LI \rightarrow \text{colim } L$ whose colimit coprojections are total. If $\epsilon: L \Rightarrow L$ is a natural transformation such that each component is a restriction idempotent, then $\text{colim } \epsilon$ is also a restriction idempotent.*

$$\begin{array}{ccc}
LI & \xrightarrow{p_I} & \text{colim } L \\
\epsilon_I \downarrow & & \downarrow \text{colim } \epsilon \\
LI & \xrightarrow{p_I} & \text{colim } L
\end{array}$$

Proof. By the fact $\overline{p_I} = 1$ and $\epsilon_I = \overline{\epsilon_I}$, we have

$$\overline{\text{colim } \epsilon} \circ p_I = p_I \circ \overline{\text{colim } \epsilon} \circ \overline{p_I} = p_I \circ \overline{p_I} \circ \overline{\epsilon_I} = p_I \circ \overline{p_I} \circ \epsilon_I = p_I \circ \overline{\epsilon_I} = p_I \circ \epsilon_I.$$

Therefore, $\text{colim } \epsilon = \overline{\text{colim } \epsilon}$ by uniqueness. □

Lemma 16. *Let \mathbf{C} be a cocomplete \mathcal{M} -category, and let $H, K: \mathbf{I} \rightarrow \mathbf{C}$ be functors (with \mathbf{I} small). Suppose $\alpha: H \Rightarrow K$ is a natural transformation such that for each $I \in \mathbf{I}$, α_I is in $\mathcal{M}_{\mathbf{C}}$ and all naturality squares are pullbacks:*

$$\begin{array}{ccc}
HI & \xrightarrow{Hf} & HJ \\
\alpha_I \downarrow & & \downarrow \alpha_J \\
KI & \xrightarrow{Kf} & KJ
\end{array}$$

Then $\text{colim } \alpha$ is in $\mathcal{M}_{\mathbf{C}}$, and the following is a pullback for every $I \in \mathbf{I}$:

$$\begin{array}{ccc}
HI & \xrightarrow{p_I} & \text{colim } H \\
\alpha_I \downarrow & & \downarrow \text{colim } \alpha \\
KI & \xrightarrow{q_I} & \text{colim } K
\end{array}$$

where p_I, q_I are colimit coprojections.

Proof. Applying the inclusion $\iota: \mathbf{C} \rightarrow \text{Par}(\mathbf{C})$ gives the following commutative diagram for each $I \in \mathbf{I}$:

$$\begin{array}{ccc}
HI & \xrightarrow{(1, p_I)} & \text{colim } H \\
(1, \alpha_I) \downarrow & & \downarrow (1, \text{colim } \alpha) \\
KI & \xrightarrow{(1, q_I)} & \text{colim } K
\end{array}$$

Observe that there is a natural transformation $\beta: \iota K \Rightarrow \iota H$ whose components are given by $\beta_I = (\alpha_I, 1)$; simply apply Lemma 14 to our assumption that α_I is a pullback of α_J along Kf .

Now the fact that the inclusion preserves the colimits $(\text{colim } H, p_I)_{i \in \mathbf{I}}$ and $(\text{colim } K, q_I)_{i \in \mathbf{I}}$ implies the existence of a unique map $\text{colim } \beta = (n, g): \text{colim } K \rightarrow \text{colim } H$ making the following diagram commute for each $I \in \mathbf{I}$:

$$\begin{array}{ccc}
KI & \xrightarrow{(1, q_I)} & \text{colim } K \\
(\alpha_I, 1) \downarrow & & \downarrow (n, g) \\
HI & \xrightarrow{(1, p_I)} & \text{colim } H \\
(1, \alpha_I) \downarrow & & \downarrow (1, \text{colim } \alpha) \\
KI & \xrightarrow{(1, q_I)} & \text{colim } K
\end{array}$$

Observe that the left composite $(1, \alpha_I) \circ (\alpha_I, 1) = (\alpha_I, \alpha_I)$ is the component at I of a natural transformation $\epsilon: \iota K \Rightarrow \iota K$ whose components are restriction idempotents. Therefore, by Lemma 15, the composite on the right $(1, \text{colim } \alpha) \circ (n, g) = (n, (\text{colim } \alpha)g)$ must be a restriction idempotent, and so $n = (\text{colim } \alpha)g$.

On the other hand, the composite $(\alpha_I, 1) \circ (1, \alpha_I) = (1, 1)$ is the component of the identity natural transformation $\gamma: \iota H \Rightarrow \iota H$ at I , and so $\text{colim } \gamma: \text{colim } H \rightarrow \text{colim } H$ must be $(1, 1)$. However, as the following diagram also commutes, we must have $(n, g) \circ (1, \text{colim } \alpha) = (1, 1)$ by uniqueness:

$$\begin{array}{ccc}
HI & \xrightarrow{(1, p_I)} & \operatorname{colim} H \\
(1, \alpha_I) \downarrow & & \downarrow (1, \operatorname{colim} \alpha) \\
KI & \xrightarrow{(1, q_I)} & \operatorname{colim} K \\
(\alpha_I, 1) \downarrow & & \downarrow (n, g) \\
HI & \xrightarrow{(1, p_I)} & \operatorname{colim} H
\end{array}$$

So we have that $(1, \operatorname{colim} \alpha) \circ (n, g) = (n, n)$ is a splitting of the restriction idempotent (n, n) , which means that $(1, \operatorname{colim} \alpha)$ is a restriction monic. Therefore $\operatorname{colim} \alpha \in \mathcal{M}_{\mathbf{C}}$, proving the first part of the lemma.

Regarding the second part of the lemma, observe that $(n, g) \circ (1, \operatorname{colim} \alpha) = (1, 1)$ implies g is an isomorphism (as $n = (\operatorname{colim} \alpha)$). Therefore, $(n, g) = (\operatorname{colim} \alpha, 1)$ and so the following diagram commutes for all $I \in \mathbf{I}$:

$$\begin{array}{ccc}
KI & \xrightarrow{(1, q_I)} & \operatorname{colim} K \\
(\alpha_I, 1) \downarrow & & \downarrow (\operatorname{colim} \alpha, 1) \\
HI & \xrightarrow{(1, p_I)} & \operatorname{colim} H
\end{array}$$

The result then follows by applying Lemma 14. \square

Lemma 17. *Let \mathbf{C} be a cocomplete \mathcal{M} -category, $H, K: \mathbf{I} \rightarrow \mathbf{C}$ functors (with \mathbf{I} small), and $\alpha: H \Rightarrow K$ a natural transformation such that each $\alpha_I \in \mathcal{M}_{\mathbf{C}}$ and all naturality squares are pullbacks (as in the previous lemma). Let $n \in \mathcal{M}_{\mathbf{C}}$, and suppose $x: \operatorname{colim} H \rightarrow X$ and $y: \operatorname{colim} K \rightarrow Y$ make the right square commute and the outer square a pullback (for all $I \in \mathbf{I}$):*

$$\begin{array}{ccccc}
HI & \xrightarrow{p_I} & \operatorname{colim} H & \xrightarrow{x} & X \\
\alpha_I \downarrow & & \downarrow \operatorname{colim} \alpha & & \downarrow n \\
KI & \xrightarrow{q_I} & \operatorname{colim} K & \xrightarrow{y} & Y
\end{array}$$

Then the right square is also a pullback.

Proof. By Lemma 14, to show that the right square is a pullback is the same as showing $(1, x) \circ (\operatorname{colim} \alpha, 1) = (\operatorname{colim} \alpha, x) = (n, 1) \circ (1, y)$ in $\operatorname{Par}(\mathbf{C})$. In other words, that the top-right square of the following diagram commutes:

$$\begin{array}{ccccc}
KI & \xrightarrow{(1, q_I)} & \operatorname{colim} K & \xrightarrow{(1, y)} & Y \\
(\alpha_I, 1) \downarrow & & \downarrow (\operatorname{colim} \alpha, 1) & & \downarrow (n, 1) \\
HI & \xrightarrow{(1, p_I)} & \operatorname{colim} H & \xrightarrow{(1, x)} & X \\
(1, \alpha_I) \downarrow & & \downarrow (1, \operatorname{colim} \alpha) & & \downarrow (1, n) \\
KI & \xrightarrow{(1, q_I)} & \operatorname{colim} K & \xrightarrow{(1, y)} & Y
\end{array}$$

Since $(\operatorname{colim} \alpha, x)$ and $(n, 1)(1, y)$ are both maps out of $\operatorname{colim} K$, it is enough to show that

$$(\operatorname{colim} \alpha, x)(1, q_I) = (n, 1)(1, y)(1, q_I)$$

for all $I \in \mathbf{I}$. But the left-hand side is equal to (α_I, xp_I) by commutativity of the top-left square, and the right-hand side is also (α_I, xp_I) by assumption. Hence the result follows. \square

Corollary 18. *If $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ is a cocomplete \mathcal{M} -category, then colimits in \mathbf{C} are stable under pullback along $\mathcal{M}_{\mathbf{C}}$ -maps.*

Proof. Let $K: \mathbf{I} \rightarrow \mathbf{C}$ be a functor, P any object in \mathbf{C} , and suppose $\mu: P \rightarrow \operatorname{colim} K$ is an $\mathcal{M}_{\mathbf{C}}$ -map. Since $\mu \in \mathcal{M}_{\mathbf{C}}$, for each $I \in \mathbf{I}$, we may take pullbacks of μ along the colimiting coprojections of $\operatorname{colim} K$, $(k_I: K_I \rightarrow \operatorname{colim} K)_{I \in \mathbf{I}}$, and these we call $\alpha_I: HI \rightarrow KI$. This gives a functor $H: \mathbf{I} \rightarrow \mathbf{C}$, which on objects, takes I to HI , and on morphisms, takes $f: I \rightarrow J$ to the unique map making all squares in the following diagram pullbacks:

$$\begin{array}{ccccc} & & p_I & & \\ & \nearrow & & \searrow & \\ HI & \xrightarrow{Hf} & HJ & \xrightarrow{p_J} & P \\ \alpha_I \downarrow & & \downarrow \alpha_J & & \downarrow \mu \\ KI & \xrightarrow{Kf} & KJ & \xrightarrow{k_J} & \operatorname{colim} K \\ & \searrow & & \nearrow & \\ & & k_I & & \end{array}$$

By construction, $(P, p_I)_{I \in \mathbf{I}}$ is a cocone in \mathbf{C} and $\alpha: H \rightarrow K$ is a natural transformation. Now let $(h_I: HI \rightarrow \operatorname{colim} H)_{I \in \mathbf{I}}$ be the colimiting coprojections of $\operatorname{colim} H$. Then by the universal property of $\operatorname{colim} H$, there exists a unique $\gamma: \operatorname{colim} H \rightarrow P$ such that $p_I = \gamma h_I$ for all $I \in \mathbf{I}$, and by the universal property of $\operatorname{colim} K$, there is a colim $\alpha: \operatorname{colim} H \rightarrow \operatorname{colim} K$ making the left square of the following diagram commute (for all $I \in \mathbf{I}$):

$$\begin{array}{ccccc} & & p_I & & \\ & \nearrow & & \searrow & \\ HI & \xrightarrow{h_I} & \operatorname{colim} H & \xrightarrow{\gamma} & P \\ \alpha_I \downarrow & & \downarrow \operatorname{colim} \alpha & & \downarrow \mu \\ KI & \longrightarrow & \operatorname{colim} K & \xlongequal{\quad} & \operatorname{colim} K \\ & \searrow & & \nearrow & \\ & & k_I & & \end{array}$$

It is easy to see that the right square commutes, and since the left square is a pullback for every $I \in \mathbf{I}$, the right square must be a pullback by Lemma 17. Therefore, because the pullback of the identity $1_{\operatorname{colim} K}$ is the identity, $P \cong \operatorname{colim} H$, and hence colimits are preserved by pullbacks along $\mathcal{M}_{\mathbf{C}}$ -maps. \square

We now show that for any small \mathcal{M} -category \mathbf{C} , the Yoneda embedding $\mathbf{y}: \mathbf{C} \rightarrow \operatorname{PSh}_{\mathcal{M}}(\mathbf{C})$ exhibits the \mathcal{M} -category of presheaves $\operatorname{PSh}_{\mathcal{M}}(\mathbf{C})$ as the free cocompletion of \mathbf{C} .

Theorem 19. (*Free cocompletion of \mathcal{M} -categories*) For any small \mathcal{M} -category \mathbf{C} and cocomplete \mathcal{M} -category \mathbf{D} , the following is an equivalence of categories:

$$(3.1) \quad (-) \circ \mathbf{y}: \mathcal{M}\mathbf{Cocomp}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \rightarrow \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D}).$$

Proof. We know that $(-) \circ \mathbf{y}: \mathbf{Cocomp}(\mathbf{PSh}(\mathbf{C}), \mathbf{D}) \rightarrow \mathbf{Cat}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories; that is, given a functor $F: \mathbf{C} \rightarrow \mathbf{D}$, there is a cocontinuous $G: \mathbf{PSh}(\mathbf{C}) \rightarrow \mathbf{D}$ such that $G\mathbf{y} \cong F$. So (3.1) will be essentially surjective on objects if this same G is an \mathcal{M} -functor.

To see that G takes monics in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ to monics in $\mathcal{M}_{\mathbf{D}}$, let $\mu: P \Rightarrow Q$ be an $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ -map. Since every presheaf is a colimit of representables, write $Q \cong \text{colim } \mathbf{y}D$, where $D: \mathbf{I} \rightarrow \mathbf{C}$ is a functor (with \mathbf{I} small). By definition of $\mu \in \mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$, for every $I \in \mathbf{I}$, there is a map $m_I: C_I \rightarrow D_I$ making the following a pullback:

$$\begin{array}{ccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \\ \mathbf{y}m_I \downarrow & & \downarrow \mu \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q \end{array}$$

(where q_I is a colimit coprojection). It follows there is a functor $C: \mathbf{I} \rightarrow \mathbf{C}$ which on objects takes I to C_I and on morphisms, takes $f: I \rightarrow J$ to the unique map Cf making the diagram below commute and the left square a pullback:

$$\begin{array}{ccccc} & & p_I & & \\ & \swarrow & & \searrow & \\ \mathbf{y}C_I & \xrightarrow{\mathbf{y}Cf} & \mathbf{y}C_J & \xrightarrow{p_J} & P \\ \mathbf{y}m_I \downarrow & & \mathbf{y}m_J \downarrow & & \downarrow \mu \\ \mathbf{y}D_I & \xrightarrow{\mathbf{y}Df} & \mathbf{y}D_J & \xrightarrow{q_J} & Q \\ & \swarrow & & \searrow & \\ & & q_I & & \end{array}$$

The fact colimits in $\mathbf{PSh}(\mathbf{C})$ are stable under pullback implies $(p_I: \mathbf{y}C_I \rightarrow P)_{I \in \mathbf{I}}$ is colimiting. Now applying G to the above diagram gives

$$(3.2) \quad \begin{array}{ccccc} & & Gp_I & & \\ & \swarrow & & \searrow & \\ G\mathbf{y}C_I & \xrightarrow{G\mathbf{y}Cf} & G\mathbf{y}C_J & \xrightarrow{Gp_J} & GP \\ G\mathbf{y}m_I \downarrow & & G\mathbf{y}m_J \downarrow & & \downarrow G\mu \\ G\mathbf{y}D_I & \xrightarrow{G\mathbf{y}Df} & G\mathbf{y}D_J & \xrightarrow{Gq_J} & GQ \\ & \swarrow & & \searrow & \\ & & Gq_I & & \end{array}$$

Since G is cocontinuous, both $(Gp_I)_{I \in \mathbf{I}}$ and $(Gq_I)_{I \in \mathbf{I}}$ are colimiting. Also, as $G\mathbf{y} \cong F$ and F is an \mathcal{M} -functor, the left square is a pullback for every pair $I, J \in \mathbf{I}$. Therefore, by Lemma 16, $G\mu$ must be in $\mathcal{M}_{\mathbf{D}}$.

Observe that the same lemma (Lemma 16) says that for every $I \in \mathbf{I}$, the outer square in (3.2) is a pullback for every $I \in \mathbf{I}$. In other words, G preserves

pullbacks of the form

$$(3.3) \quad \begin{array}{ccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \\ \mathbf{y}m_I \downarrow & & \downarrow \mu \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q \end{array}$$

Now to see that G preserves $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ -pullbacks, consider the diagram below, where the right square is an $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ -pullback and the left square is a pullback for all $I \in \mathbf{I}$:

$$\begin{array}{ccccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \cong \operatorname{colim} \mathbf{y}C & \longrightarrow & P' \\ \mathbf{y}m_I \downarrow & & \downarrow \mu & & \downarrow \mu' \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q \cong \operatorname{colim} \mathbf{y}D & \longrightarrow & Q' \end{array}$$

The result then follows by applying G to the diagram and using Lemma 17. This proves (3.1) is essentially surjective on objects.

Finally, to show that (3.1) is fully faithful, we need to show for any cocontinuous pair of \mathcal{M} -functors $F, F': \mathbf{PSh}_{\mathcal{M}}(\mathbf{C}) \rightarrow \mathbf{D}$ and $\mathcal{M}_{\mathbf{C}}$ -cartesian $\alpha: F\mathbf{y} \Rightarrow F'\mathbf{y}$, there exists a unique $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ -cartesian $\tilde{\alpha}: F \Rightarrow F'$ such that $\tilde{\alpha}\mathbf{y} = \alpha$. In other words, the following is an isomorphism of sets:

$$(-) \circ \mathbf{y}: \mathcal{M}\mathbf{Nat}(F, F') \rightarrow \mathcal{M}\mathbf{Nat}(F\mathbf{y}, F'\mathbf{y})$$

where $\mathcal{M}\mathbf{Nat}(F, F')$ are the \mathcal{M} -cartesian natural transformations between F and F' . However, this condition may be reformulated as follows:

$$(3.4) \quad \begin{array}{l} \text{For all natural transformations } \tilde{\alpha}: F \Rightarrow F', \tilde{\alpha} \text{ is } \mathcal{M}_{\mathbf{PSh}(\mathbf{C})}\text{-cartesian if} \\ \tilde{\alpha}\mathbf{y}: F\mathbf{y} \Rightarrow F'\mathbf{y} \text{ is } \mathcal{M}_{\mathbf{C}}\text{-cartesian.} \end{array}$$

To see that these two statements are equivalent, observe that the second statement amounts to the following diagram being a pullback in \mathbf{Set} :

$$\begin{array}{ccc} \mathcal{M}\mathbf{Nat}(F, F') & \longrightarrow & \mathcal{M}\mathbf{Nat}(F\mathbf{y}, F'\mathbf{y}) \\ \downarrow & & \downarrow \\ \mathbf{Nat}(F, F') & \xrightarrow{(-) \circ \mathbf{y}} & \mathbf{Nat}(F\mathbf{y}, F'\mathbf{y}) \end{array}$$

where $\mathbf{Nat}(F, F')$ is the set of natural transformations between F and F' . However, as the bottom function is an isomorphism (ordinary free cocompletion), the top must also be an isomorphism and hence the two statements are equivalent. Therefore, we show (3.1) is fully faithful by proving (3.4).

So let $\mu: P \Rightarrow Q$ be an $\mathcal{M}_{\text{PSh}(\mathbf{C})}$ -map, and recall that the left square (diagram below) is a pullback for every $I \in \mathbf{I}$ as F preserves $\mathcal{M}_{\text{PSh}(\mathbf{C})}$ -pullbacks:

$$(3.5) \quad \begin{array}{ccccc} F\mathbf{y}C_I & \xrightarrow{Fp_I} & FP & \xrightarrow{\tilde{\alpha}_P} & F'P \\ F\mathbf{y}m_I \downarrow & & \downarrow F\mu & & \downarrow F'\mu \\ F\mathbf{y}D_I & \xrightarrow{Fq_I} & FQ & \xrightarrow{\tilde{\alpha}_Q} & F'Q \end{array}$$

To show that the right square is a pullback, we will show that the outer square is a pullback for every $I \in \mathbf{I}$ and apply Lemma 17. Now by naturality of $\tilde{\alpha}$, this outer square is the outer square of the following diagram:

$$\begin{array}{ccccc} F\mathbf{y}C_I & \xrightarrow{\tilde{\alpha}_{\mathbf{y}C_I}} & F'\mathbf{y}C_I & \xrightarrow{F'p_I} & F'P \\ F\mathbf{y}m_I \downarrow & & \downarrow F'\mathbf{y}m_I & & \downarrow F'\mu \\ F\mathbf{y}D_I & \xrightarrow{\tilde{\alpha}_{\mathbf{y}D_I}} & F'\mathbf{y}D_I & \xrightarrow{F'q_I} & F'Q \end{array}$$

But $\tilde{\alpha} \circ \mathbf{y}$ being $\mathcal{M}_{\mathbf{C}}$ -cartesian implies the left square is a pullback, and the right square is also a pullback by the fact F' preserves pullbacks of the form (3.3). Therefore, by Lemma 17, each square on the right of (3.5) is a pullback, and so $\tilde{\alpha}$ is $\mathcal{M}_{\text{PSh}(\mathbf{C})}$ -cartesian. Hence, $(-) \circ \mathbf{y}: \mathcal{M}\text{Cocomp}(\text{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \rightarrow \mathcal{M}\text{Cat}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories. \square

3.4. Cocompletion of restriction categories. Earlier, we explored the notion of cocomplete \mathcal{M} -category. Now, by the fact $\mathcal{M}\text{Cat}$ and \mathbf{rCat}_s are 2-equivalent, it makes sense to define a restriction category to be cocomplete in such a way that $\text{Par}(\mathbf{C})$ will be cocomplete as a restriction category if and only if \mathbf{C} is cocomplete as an \mathcal{M} -category.

Definition 20. A restriction category \mathbf{X} is cocomplete if it is split, its subcategory $\text{Total}(\mathbf{X})$ is cocomplete, and the inclusion $\text{Total}(\mathbf{X}) \hookrightarrow \mathbf{X}$ preserves colimits. A restriction functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ is cocontinuous if $\text{Total}(F): \text{Total}(\mathbf{X}) \rightarrow \text{Total}(\mathbf{Y})$ is cocontinuous. We denote by $\mathbf{rCocomp}$ the 2-category of cocomplete restriction categories, cocontinuous restriction functors and restriction transformations.

Observe that for any cocomplete restriction category \mathbf{X} , $\mathcal{M}\text{Total}(\mathbf{X})$ is a cocomplete \mathcal{M} -category since $\text{Total}(\mathbf{X})$ is cocomplete and $\text{Total}(\mathbf{X}) \hookrightarrow \mathbf{X} \cong \text{Par}(\mathcal{M}\text{Total}(\mathbf{X}))$ preserves colimits. We now give examples of cocomplete restriction categories.

Example 21. For each class of examples from Example 10, $\text{Par}(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$ is a cocomplete restriction category. In particular, the restriction category of sets and partial functions \mathbf{Set}_p is a cocomplete restriction category since $\mathbf{Set}_p = \text{Par}(\mathcal{E}, \mathcal{M}_{\mathcal{E}})$, where $\mathcal{E} = \mathbf{Set}$ and $\mathcal{M}_{\mathcal{E}}$ are the injective functions.

Also note that since the \mathcal{M} -category $(\mathbf{Ab}, \mathcal{M}_{\mathbf{Ab}})$ of abelian groups and group monomorphisms is not cocomplete as an \mathcal{M} -category, $\text{Par}(\mathbf{Ab}, \mathcal{M}_{\mathbf{Ab}})$ is also not a cocomplete restriction category.

We know that for any small \mathcal{M} -category \mathbf{C} , $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ is a cocomplete \mathcal{M} -category, and furthermore, $\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}))$ is a cococomplete restriction category. In particular, the split restriction category $\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$ is a cococomplete restriction category for any small restriction category \mathbf{X} . We now show that the Cockett and Lack embedding below [3, p. 252]

$$(3.6) \quad \mathbf{X} \xrightarrow{J} \mathbf{K}_r(\mathbf{X}) \xrightarrow{\Phi_{\mathbf{K}_r(\mathbf{X})}} \mathbf{Par}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))) \xrightarrow{\mathbf{Par}(\mathbf{y})} \mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$$

exhibits this split restriction category $\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$ as the free restriction cocompletion of any small restriction category \mathbf{X} .

Theorem 22. (*Free cocompletion of restriction categories*) *For any small restriction category \mathbf{X} and cococomplete restriction category \mathcal{E} , the following is an equivalence of categories:*

$$(-) \circ (3.6): \mathbf{rCocomp}(\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X})))) , \mathcal{E}) \rightarrow \mathbf{rCat}(\mathbf{X}, \mathcal{E})$$

where (3.6) is the Cockett and Lack embedding.

Proof. First note that $\mathcal{E} \cong \mathbf{Par}(\mathbf{D})$ for some cocomplete \mathcal{M} -category \mathbf{D} (as \mathcal{E} is split), and that

$$\mathbf{rCocomp}(\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})), \mathbf{Par}(\mathbf{D})) \simeq \mathcal{M}\mathbf{Cocomp}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D})$$

since \mathbf{Par} and $\mathcal{M}\mathbf{Total}$ are 2-equivalences. Therefore,

$$(-) \circ \mathbf{Par}(\mathbf{y}): \mathbf{rCocomp}(\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})), \mathcal{E}) \rightarrow \mathbf{rCat}(\mathbf{Par}(\mathbf{C}), \mathcal{E})$$

is an equivalence since

$$(-) \circ \mathbf{y}: \mathcal{M}\mathbf{Cocomp}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \rightarrow \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D})$$

is an equivalence (free cocompletion of \mathcal{M} -categories). Therefore the following composite is an equivalence:

$$\begin{array}{c} \mathbf{rCocomp}(\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X})))) , \mathcal{E} \\ \downarrow (-) \circ \mathbf{Par}(\mathbf{y}) \\ \mathbf{rCocomp}(\mathbf{Par}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X})))) , \mathcal{E} \\ \downarrow (-) \circ \Phi_{\mathbf{K}_r(\mathbf{X})} \circ J \\ \mathbf{rCat}(\mathbf{X}, \mathcal{E}) \end{array}$$

as $\Phi_{\mathbf{K}_r(\mathbf{X})}$ is an isomorphism and J is the unit of the biadjunction $i \dashv \mathbf{K}_r$ at \mathbf{X} . \square

4. RESTRICTION PRESHEAVES

We have just seen that for any small restriction category \mathbf{X} , the Cockett-Lack embedding (3.6) exhibits the restriction category $\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$ as a free cocompletion of \mathbf{X} . However, this formulation of free cocompletion seems rather complex compared to the fact both $\mathbf{PSh}(\mathbf{C})$ and $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ were the free cocompletions of ordinary categories and \mathcal{M} -categories respectively.

In this section, we give an alternate simpler definition in terms of a restriction category of *restriction presheaves* $\mathbf{PSh}_r(\mathbf{X})$. We shall see that $\mathbf{PSh}_r(\mathbf{X})$ is a full subcategory of $\mathbf{PSh}(\mathbf{X})$ and that the Yoneda embedding factors through a restriction functor $\mathbf{y}_r: \mathbf{X} \rightarrow \mathbf{PSh}_r(\mathbf{X})$. Finally, we show that the category of restriction presheaves $\mathbf{PSh}_r(\mathbf{X})$ is equivalent to $\mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$, so that it gives another way of describing free cocompletion in the restriction setting.

Definition 23. (Restriction presheaf) Let \mathbf{X} be a restriction category. A *restriction presheaf* on \mathbf{X} is a presheaf $P: \mathbf{X}^{\text{op}} \rightarrow \mathbf{Set}$ together with assignments

$$PA \rightarrow \mathbf{X}(A, A), \quad x \mapsto \bar{x}$$

where \bar{x} is a restriction idempotent satisfying the following three axioms:

- (A1) $x \cdot \bar{x} = x$;
- (A2) $\overline{x \cdot \bar{f}} = \bar{x} \circ \bar{f}$, where $\bar{f}: A \rightarrow A$ is a restriction idempotent in \mathbf{X} ;
- (A3) $\bar{x} \circ g = g \circ \overline{x \cdot g}$, where $g: B \rightarrow A$ in \mathbf{X} .

The notation $x \cdot \bar{x}$ denotes the element $P(\bar{x})(x) \in PA$ [10, p. 25]. We call the assignments $x \mapsto \bar{x}$ the *restriction structure* on P .

Unlike the restriction structure on a restriction category, the restriction structure on any restriction presheaf is unique, due to the following lemma.

Lemma 24. Let \mathbf{X} be a restriction category and $P: \mathbf{X}^{\text{op}} \rightarrow \mathbf{Set}$ a presheaf. Suppose P has two restriction structures given by $x \mapsto \bar{x}$ and $x \mapsto \tilde{x}$. Then $\bar{x} = \tilde{x}$ for all $A \in \mathbf{X}$ and $x \in PA$.

Proof. By the fact \bar{x} and \tilde{x} are restriction idempotents and using (A1),(A2),

$$\bar{x} = \overline{x \cdot \tilde{x}} = \bar{x} \circ \tilde{x} = \tilde{x} \circ \bar{x} = \widetilde{x \cdot \bar{x}} = \tilde{x}. \quad \square$$

We also have the following analogues of basic results of restriction categories.

Lemma 25. Suppose P is a restriction presheaf on a restriction category \mathbf{X} , and let $A \in \mathbf{X}$, $x \in PA$ and $g: B \rightarrow A$. Then

- (1) $\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot \bar{g}}$;
- (2) $\bar{x} \circ g = \overline{x \cdot g}$.

Proof. (1) By (R2), (A2) and (R1),

$$\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot \bar{g}} \circ \bar{g} = \overline{(x \cdot g) \cdot \bar{g}} = \overline{x \cdot (g \circ \bar{g})} = \overline{x \cdot \bar{g}}.$$

(2) By (A3), (R3) and the previous result,

$$\overline{\bar{x} \circ g} = \overline{g \circ \overline{x \cdot g}} = \bar{g} \circ \overline{x \cdot \bar{g}} = \overline{x \cdot \bar{g}}. \quad \square$$

Definition 26. (Category of restriction presheaves) Let \mathbf{X} be a restriction category. The category of restriction presheaves on \mathbf{X} , $\mathbf{PSh}_r(\mathbf{X})$, has the following data:

Objects: Restriction presheaves;

Morphisms: Arbitrary natural transformations $\alpha: P \Rightarrow Q$;

Restriction: The restriction of $\alpha: P \Rightarrow Q$ is the natural transformation $\bar{\alpha}: P \Rightarrow P$, given componentwise by

$$\bar{\alpha}_A(x) = x \cdot \overline{\alpha_A(x)}$$

for every $A \in \mathbf{X}$ and $x \in PA$.

Note that $\bar{\alpha}$ is natural since

$$\bar{\alpha}_B(x \cdot f) = x \cdot \left(f \circ \overline{\alpha_B(x \cdot f)} \right) = x \cdot \left(f \circ \overline{\alpha_A(x)} \cdot f \right) = x \cdot \left(\overline{\alpha_A(x)} \circ f \right) = \bar{\alpha}_A(x) \cdot f$$

for all $f: B \rightarrow A$. The restriction axioms are easy to check.

Observe that $\mathbf{PSh}_r(\mathbf{X})$ is a full subcategory of $\mathbf{PSh}(\mathbf{X})$, as the restriction structure on any presheaf, if it exists, must be unique. Also, if \mathbf{X} is a restriction category, then each representable $\mathbf{X}(-, A)$ has a restriction structure given by sending $f \in \mathbf{X}(B, A)$ to $\bar{f} \in \mathbf{X}$. In particular, this implies that the Yoneda embedding $\mathbf{y}: \mathbf{X} \rightarrow \mathbf{PSh}(\mathbf{X})$ factors as a unique functor $\mathbf{y}_r: \mathbf{X} \rightarrow \mathbf{PSh}_r(\mathbf{X})$.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{y}_r} & \mathbf{PSh}_r(\mathbf{X}) \\ & \searrow \mathbf{y} & \downarrow \\ & & \mathbf{PSh}(\mathbf{X}) \end{array}$$

Lemma 27. *For any restriction category \mathbf{X} , the functor $\mathbf{y}_r: \mathbf{X} \rightarrow \mathbf{PSh}_r(\mathbf{X})$ is a restriction functor.*

Proof. Let $f: A \rightarrow B$ be a map in \mathbf{X} . Then for all $X \in \mathbf{X}$ and $x \in \mathbf{X}(X, A)$, we have

$$\overline{\mathbf{y}_r f}_X(x) = x \cdot \overline{(\mathbf{y}_r f)_X(x)} = x \cdot \overline{f \circ x} = x \circ \overline{f \circ x} = \bar{f} \circ x = (\mathbf{y}_r \bar{f})_X(x)$$

and so \mathbf{y}_r is a restriction functor. \square

We can characterise the total maps in $\mathbf{PSh}_r(\mathbf{X})$ as those which are restriction preserving, due to the following proposition.

Proposition 28. *A map $\alpha: P \Rightarrow Q$ is total in $\mathbf{PSh}_r(\mathbf{X})$ if and only if $\overline{\alpha_A(x)} = \bar{x}$ for all $A \in \mathbf{X}$ and $x \in PA$.*

Proof. Suppose $\alpha: P \Rightarrow Q$ is total in $\mathbf{PSh}_r(\mathbf{X})$. Then $\bar{\alpha}_A(x) = 1_{PA}(x) = x$, or $x \cdot \overline{\alpha_A(x)} = x$. But this implies $\bar{x} \leq \overline{\alpha_A(x)}$ since

$$\bar{x} = \overline{x \cdot \overline{\alpha_A(x)}} = \bar{x} \circ \overline{\alpha_A(x)} = \overline{\alpha_A(x)} \circ \bar{x}$$

On the other hand, $\overline{\alpha_A(x)} \leq \bar{x}$ as

$$\overline{\alpha_A(x)} = \overline{\alpha_A(x \cdot \bar{x})} = \overline{\alpha_A(x)} \cdot \bar{x} = \overline{\alpha_A(x)} \circ \bar{x} = \bar{x} \circ \overline{\alpha_A(x)}$$

Therefore, α in $\mathbf{PSh}_r(\mathbf{X})$ is total if and only if α preserves restrictions. \square

The restriction presheaf category has one more important property.

Proposition 29. *Let \mathbf{X} be a restriction category. Then $\mathbf{PSh}_r(\mathbf{X})$ is a split restriction category.*

Proof. Let $\bar{\alpha}: P \Rightarrow P$ be a restriction idempotent in $\mathbf{PSh}_r(\mathbf{X})$. Since all idempotents in $\mathbf{PSh}(\mathbf{X})$ split, we may write $\bar{\alpha} = \mu\rho$ for some maps $\mu: Q \Rightarrow P$ and $\rho: P \Rightarrow Q$ such that $\rho\mu = 1$. Componentwise, we may take μ_A to be the inclusion $QA \hookrightarrow PA$ and $QA = \{x \in PA \mid \bar{\alpha}_A(x) = x\}$. Therefore, to show $\mathbf{PSh}_r(\mathbf{X})$ is split, it is enough to show that Q is a restriction presheaf. However, P is a restriction presheaf and Q is a subfunctor of P . Therefore, imposing the restriction

structure of P onto Q will make Q a restriction presheaf. Hence $\mathbf{PSh}_r(\mathbf{X})$ is a split restriction category. \square

Before moving onto the main theorems in this section, let us recall the split restriction category $\mathbf{K}_r(\mathbf{X})$, whose objects are pairs (A, e) (with e a restriction idempotent on $A \in \mathbf{X}$). Also recall the unit of the biadjunction $i \dashv \mathbf{K}_r$ at \mathbf{X} , $J: \mathbf{X} \rightarrow \mathbf{K}_r(\mathbf{X})$, which sends objects A to $(A, 1_A)$ and morphisms $f: A \rightarrow B$ to $f: (A, 1_A) \rightarrow (B, 1_B)$.

Proposition 30. $\mathbf{PSh}_r(\mathbf{X})$ and $\mathbf{PSh}_r(\mathbf{K}_r(\mathbf{X}))$ are equivalent as restriction categories.

Proof. It is well-known that the functor $(-) \circ J^{\text{op}}: \mathbf{PSh}(\mathbf{K}_r(\mathbf{X})) \rightarrow \mathbf{PSh}(\mathbf{X})$ is an equivalence. Therefore, the result will follow if we can show this functor restricts back to an equivalence between $\mathbf{PSh}_r(\mathbf{K}_r(\mathbf{X}))$ and $\mathbf{PSh}_r(\mathbf{X})$. In other words, showing that the restriction of $(-) \circ J^{\text{op}}$ to $\mathbf{PSh}_r(\mathbf{K}_r(\mathbf{X}))$ sends objects to restriction presheaves, is essentially surjective on objects and is a restriction functor.

$$\begin{array}{ccc} \mathbf{PSh}_r(\mathbf{K}_r(\mathbf{X})) & \longrightarrow & \mathbf{PSh}_r(\mathbf{X}) \\ \downarrow & & \downarrow \\ \mathbf{PSh}(\mathbf{K}_r(\mathbf{X})) & \xrightarrow{(-) \circ J^{\text{op}}} & \mathbf{PSh}(\mathbf{X}) \end{array}$$

So let P be a restriction presheaf on $\mathbf{K}_r(\mathbf{X})$. Then PJ^{op} will be a restriction presheaf on \mathbf{X} if we define the restriction on $x \in (PJ^{\text{op}})(A) = P(A, 1_A)$ to be the same as in $P(A, 1_A)$ for all $A \in \mathbf{X}$. Also, if $\bar{\alpha}: P \Rightarrow P$ is a restriction idempotent, then

$$(\bar{\alpha} \circ J^{\text{op}})_A(x) = \bar{\alpha}_{(A, 1_A)}(x) = x \cdot \overline{\alpha_{(A, 1_A)}(x)} = x \cdot \overline{(\alpha \circ J^{\text{op}})_A(x)} = \left(\overline{\alpha \circ J^{\text{op}}} \right)_A(x)$$

implies $(-) \circ J^{\text{op}}$ is a restriction functor. Therefore, all that remains is to show essential surjectivity.

Let Q be a restriction presheaf on \mathbf{X} , and define a presheaf Q' on $\mathbf{K}_r(\mathbf{X})$ as follows:

Objects: $(A, e) \mapsto \{x \in QA \mid x \cdot e = x\}$;

Morphisms: $f: (A, e) \rightarrow (A', e') \mapsto Qf$.

A quick check will show that $Q' \circ J^{\text{op}} = Q$. Then to make Q' a restriction presheaf, observe that because $x \in Q'(A, e) \subseteq QA$ and Q is a restriction presheaf, there exists a restriction idempotent \bar{x} associated to x . Therefore, define the restriction structure on Q' to be $x \mapsto \bar{x}$. This then will satisfy the restriction presheaf axioms, making Q' a restriction presheaf. Hence, $(-) \circ J^{\text{op}}: \mathbf{PSh}(\mathbf{K}_r(\mathbf{X})) \rightarrow \mathbf{PSh}(\mathbf{X})$ is essentially surjective on objects, and so $\mathbf{PSh}_r(\mathbf{X})$ and $\mathbf{PSh}_r(\mathbf{K}_r(\mathbf{X}))$ are equivalent. \square

Before proving that $\text{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\text{Total}(\mathbf{K}_r(\mathbf{X}))))$ and $\mathbf{PSh}_r(\mathbf{X})$ are, in fact, equivalent as restriction categories, we give the following lemma.

Lemma 31. Let \mathbf{C} be a category and let m be a monic in \mathbf{C} . Suppose the following is a pullback:

$$\begin{array}{ccc}
D & \xrightarrow{g} & A \\
n \downarrow & & \downarrow m \\
B & \xrightarrow{f} & C
\end{array}$$

Then n is an isomorphism if and only if $f = mh$ for some $h: B \rightarrow A \in \mathbf{C}$.

Proof. (\Rightarrow) Take $h = gn^{-1}$. (\Leftarrow) Consider maps $1_B: B \rightarrow B$ and $h: B \rightarrow A$ and use the fact the square is a pullback. \square

We now give the following equivalence of \mathcal{M} -categories.

Theorem 32. *Suppose \mathbf{C} is an \mathcal{M} -category. Then $\mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ and $\text{PSh}_{\mathcal{M}}(\mathbf{C})$ are equivalent.*

Proof. We adopt the following approach. First, find functors $F: \text{PSh}(\mathbf{C}) \rightarrow \text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ and $G: \text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C}))) \rightarrow \text{PSh}(\mathbf{C})$, and natural isomorphisms $\eta: 1 \Rightarrow GF$ and $\epsilon: FG \Rightarrow 1$. We then show that F and G are in fact \mathcal{M} -functors. (Note that η and ϵ must necessarily be \mathcal{M} -cartesian).

So let P be a presheaf on \mathbf{C} , and define F on objects as follows. If $X \in \text{Par}(\mathbf{C})$, then $(FP)(X)$ is the set of equivalence classes

$$(FP)(X) = \{(m, f) \mid m: Y \rightarrow X \in \mathcal{M}_{\mathbf{C}}, f \in PY\}$$

where $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ such that $n = m\varphi$ and $g = f \cdot \varphi$. To define FP on morphisms, given $(n, g): Z \rightarrow X$ in $\text{Par}(\mathbf{C})$ and an element $(m, f) \in (FP)(X)$, define

$$\left((FP)(n, g)\right)(m, f) = (nm', f \cdot g')$$

where (m', g') is the pullback of (m, g) , as in:

$$\begin{array}{ccc}
\bullet & \xrightarrow{g'} & \bullet \\
m' \downarrow & & \downarrow m \\
\bullet & \xrightarrow{g} & \bullet
\end{array}$$

We shall sometimes denote the above informally as $(m, f) \cdot (n, g)$. Then defining the restriction on each $(m, f) \in (FP)(X)$ to be (m, m) makes $FP: \text{Par}(\mathbf{C})^{\text{op}} \rightarrow \mathbf{Set}$ a restriction presheaf. This defines F on objects.

Now suppose $\alpha: P \Rightarrow Q$ is a morphism in $\text{PSh}(\mathbf{C})$. Define $F\alpha: FP \rightarrow FQ$ componentwise as follows:

$$(F\alpha)_X(m, f) = (m, \alpha_{\text{dom } m}(f)).$$

Then $F\alpha$ is natural (by naturality of α) and also total, making F a functor from $\text{PSh}(\mathbf{C})$ to $\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$. We now give the data for the functor G from $\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ to $\text{PSh}(\mathbf{C})$.

Let P be a restriction presheaf on $\text{Par}(\mathbf{C})$, and define $GP: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ as follows. If $X \in \mathbf{C}$, then

$$(GP)(X) = \{x \mid x \in PX, \bar{x} = (1, 1)\}.$$

And if $f: Z \rightarrow X$ is an arrow in \mathbf{C} , define

$$(GP)(f) = P(1, f).$$

Note that $(GP)(f)$ is well-defined since for every $x \in (GP)(X)$,

$$\overline{P(1, f)(x)} = \overline{x \cdot (1, f)} = \overline{\bar{x} \circ (1, f)} = 1,$$

and so $(GP)(f)$ is a function from $(GP)(X)$ to $(GP)(Z)$.

Finally, if $\alpha: P \Rightarrow Q$ is a total map in $\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C}))$, define $G\alpha: GP \Rightarrow GQ$ componentwise by

$$(G\alpha)_X(x) = \alpha_X(x)$$

for every $X \in \mathbf{C}$ and $x \in (GP)(X)$. Again, to see that $G\alpha$ is well-defined, note that α total implies $\overline{\alpha_X(x)} = \bar{x} = 1$ (Proposition 28) and so $\alpha_X(x) \in (GQ)(X)$. This makes G a functor from $\mathbf{Total}(\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C})))$ to $\mathbf{PSh}(\mathbf{C})$. The next step is defining isomorphisms $\eta: 1 \Rightarrow GF$ and $\epsilon: FG \Rightarrow 1$.

To define η , we need to give components for every presheaf P on \mathbf{C} , and this involves giving isomorphisms $(\eta_P)_X: PX \rightarrow (GFP)(X)$. But $(GFP)(X) = \{(1, f) \mid f \in PX\}$. Therefore, defining $(\eta_P)_X(f) = (1, f)$ makes η an isomorphism, and naturality is easy to check.

Similarly, to define ϵ , we need to define isomorphisms $(\epsilon_P)_X: (FGP)(X) \rightarrow PX$ for every restriction presheaf P on $\mathbf{Par}(\mathbf{C})$ and object $X \in \mathbf{Par}(\mathbf{C})$. Since

$$(FGP)(X) = \{(m, f) \mid m: Y \rightarrow X \in \mathcal{M}_{\mathbf{C}}, f \in PY, \bar{f} = (1, 1)\},$$

define $(\epsilon_P)_X(m, f) = f \cdot (m, 1)$. Its inverse $(\epsilon_P)_X^{-1}: PX \rightarrow (FGP)(X)$ is then given by

$$(\epsilon_P)_X^{-1}(x) = (n, x \cdot (1, n))$$

where $\bar{x} = (n, n)$ (as P is a restriction presheaf on $\mathbf{Par}(\mathbf{C})$). Checking the naturality of ϵ is again straightforward. All that remains is to show that both $F: \mathbf{PSh}_{\mathcal{M}}(\mathbf{C}) \rightarrow \mathcal{M}\mathbf{Total}(\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C})))$ and $G: \mathcal{M}\mathbf{Total}(\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C}))) \rightarrow \mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ are \mathcal{M} -functors. However, as F and G are equivalences in \mathbf{Cat} , they necessarily preserve limits, and so all this will involve is showing that they preserve \mathcal{M} -maps. That is, $F\mu$ is a restriction monic in $\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C}))$ for all $\mu \in \mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$, and that $G\mu$ is in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ for all restriction monics $\mu \in \mathbf{PSh}_r(\mathbf{Par}(\mathbf{C}))$.

So let $\mu: P \Rightarrow Q$ be in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$. To show $F\mu$ is a restriction monic, we need to show $F\mu$ is the equaliser of 1 and some restriction idempotent $\alpha: FQ \Rightarrow FQ$. To define this α , let $X \in \mathbf{Par}(\mathbf{C})$ and $(n, g) \in (FQ)(X)$ (where $n: Z \rightarrow X$). Now as $g \in QZ$, there exists a corresponding natural transformation $\hat{g}: \mathbf{y}Z \Rightarrow Q$ (Yoneda). However, as μ is in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$, there exists an $m_g: B \rightarrow Z$ in $\mathcal{M}_{\mathbf{C}}$ making the following a pullback:

$$\begin{array}{ccc} \mathbf{y}B & \longrightarrow & P \\ \mathbf{y}m_g \downarrow & & \downarrow \mu \\ \mathbf{y}Z & \xrightarrow{\hat{g}} & Q \end{array}$$

So define α by its components as follows,

$$\alpha_X(n, g) = (nm_g, g \cdot m_g).$$

It is then not difficult to show this α is well-defined, is a natural transformation and is a restriction idempotent.

Now to show that $F\mu$ equalises 1 and α , we need to show $(F\mu)_X: (FP)(X) \rightarrow (FQ)(X)$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in **Set** for all $X \in \mathbf{Par}(\mathbf{C})$. In other words, that $(F\mu)_X$ is injective, and that:

(4.1)

$(n, g) \in (FQ)(X)$ satisfies $(n, g) = (F\mu)_X(m, f) = (m, \mu_{\text{dom } m}(f))$ for some $(m, f) \in (FP)(X)$ if and only if $\alpha_X(n, g) = (n, g)$.

To show $(F\mu)_X$ is injective, suppose $(F\mu)_X(m, f) = (F\mu)_X(m', f')$, or equivalently, $(m, \mu_{\text{dom } m}(f)) = (m', \mu_{\text{dom } m'}(f'))$. That is, there exists an isomorphism φ such that $m' = m\varphi$ and $\mu_{\text{dom } m'}(f') = \mu_{\text{dom } m}(f) \cdot \varphi$. But the naturality of μ implies $\mu_{\text{dom } m'}(f \cdot \varphi) = \mu_{\text{dom } m}(f) \cdot \varphi = \mu_{\text{dom } m'}(f')$. Therefore, as μ is monic, we must have $f \cdot \varphi = f'$. Hence $(m, f) = (m', f')$, and so $(F\mu)_X$ is injective.

To prove (4.1), let $(n, g) \in (FQ)(X)$ and suppose $\mu_X(n, g) = (n, g)$. That is, $(nm_g, g \cdot m_g) = (n, g)$, or that m_g is an isomorphism. Now m_g is an isomorphism if and only if $\mathbf{y}m_g$ is an isomorphism, and by Lemma 31, $\mathbf{y}m_g$ is an isomorphism if and only if $\hat{g} = \mu\hat{h}$ for some $\hat{h}: \mathbf{y}Z \rightarrow P$:

$$\begin{array}{ccc} \mathbf{y}B & \xrightarrow{\quad} & P \\ \mathbf{y}m_g \downarrow & \nearrow \hat{h} & \downarrow \mu \\ \mathbf{y}Z & \xrightarrow{\hat{g}} & Q \end{array}$$

But by Yoneda, the statement $\hat{g} = \mu\hat{h}$ is equivalent to the statement that $g = \mu_Z(h)$ for some $h \in PZ$, which is the same as saying $(n, g) = (n, \mu_Z(h)) = (F\mu)_X(n, h)$, with $(n, h) \in (FP)(X)$. Therefore, $(F\mu)_X$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in **Set** for all $X \in \mathbf{Par}(\mathbf{C})$, and hence, $F\mu$ equalises 1 and α .

Now to see that G is also an \mathcal{M} -functor, let $\mu: P \Rightarrow Q$ be a restriction monic in $\mathbf{PSh}_r(\mathbf{Par}(\mathbf{C}))$. To show $G\mu$ is in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$, we need to show for any given $\hat{\theta}: \mathbf{y}C \Rightarrow Q$, there exists a monic $m: D \rightarrow C$ in $\mathcal{M}_{\mathbf{C}}$ and a map $\hat{\delta}: \mathbf{y}D \Rightarrow P$ making the following a pullback:

$$\begin{array}{ccc} \mathbf{y}D & \xrightarrow{\hat{\delta}} & GP \\ \mathbf{y}m \downarrow & & \downarrow G\mu \\ \mathbf{y}C & \xrightarrow{\hat{\theta}} & GQ \end{array}$$

Here we make two observations. First, commutativity says m and δ must satisfy $G\mu \circ \hat{\delta} = \hat{\theta} \circ \mathbf{y}m$. On the other hand, Yoneda tells us that $\hat{\theta} \circ \mathbf{y}m = \widehat{\theta \cdot m}$ and $G\mu \circ \hat{\delta} = \widehat{(G\mu)_D(\delta)}$, where $\theta \in QC$ and $\delta \in PD$ are the unique transposes of $\hat{\theta}$ and $\hat{\delta}$ respectively. Therefore, m and δ must satisfy the following condition:

$$(4.2) \quad (G\mu)_D(\delta) = \theta \cdot_{GQ} m.$$

That is, $\mu_D(\delta) = \theta \cdot_Q (1, m)$. Secondly, m and δ must make the following a pullback in **Set** (for all objects $X \in \mathbf{C}$):

$$\begin{array}{ccc}
\mathbf{C}(X, D) & \xrightarrow{\hat{\delta}_X = \delta \cdot_{GP} (-)} & (GP)(X) \\
m \circ (-) \downarrow & & \downarrow (G\mu)_X \\
\mathbf{C}(X, C) & \xrightarrow{\hat{\theta}_X = \theta \cdot_{GQ} (-)} & (GQ)(X)
\end{array}$$

In other words, for any $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_{GQ} f = (G\mu)_X(x)$ (i.e., such that $\theta \cdot_Q (1, f) = \mu_X(x)$), there exists a unique $g \in \mathbf{C}(X, D)$ such that

$$(4.3) \quad \delta \cdot_{GP} g = x, \text{ and } mg = f.$$

Alternatively, $\delta \cdot_P (1, g) = x$ and $mg = f$. To find m , note that because μ is a restriction monic, there exists a ρ such that $\mu\rho = \bar{\rho}$ and $\overline{\rho\mu} = 1$. Since $\theta \in QC$, applying ρ_C to θ and then taking its restriction gives $\overline{\rho_C(\theta)} = (m, m)$ for some $m \in \mathcal{M}_C$. This gives us m .

To define δ , observe that $P(1, m)$ is a function from PC to PD . So define

$$\delta = \rho_C(\theta) \cdot_P (1, m).$$

Then $\delta \in (GP)(D)$ since

$$\bar{\delta} = \overline{\rho_C(\theta)} \circ (1, m) = \overline{(m, m)} \circ (1, m) = \overline{(1, m)} = (1, 1).$$

So all that remains is to show m and δ satisfy (4.2) and (4.3). To show m and δ satisfy (4.2), one simply substitutes the given values into the equation, using the fact $\mu\rho = \bar{\rho}$. To see that (4.3) is also satisfied, suppose there exists an $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_P (1, f) = \mu_X(x)$. Then applying ρ_X to both sides gives

$$\rho_C(\theta) \cdot_P (1, f) = x$$

since $\rho\mu = 1$. We need to show there exists a g such that $mg = f$ and $\delta \cdot_P (1, g) = x$. But $mg = f$ implies

$$x = \rho_C(\theta) \cdot_P (1, f) = \rho_C(\theta) \cdot_P (1, mg) = \rho_C(\theta) \cdot_P (1, m) \cdot_P (1, g) = \delta \cdot_P (1, g)$$

Therefore, we just need to find g .

Consider the composite $(m, m) \circ (1, f) = (m', mf')$, where (m', f') is the pullback of (m, f) :

$$\begin{array}{ccc}
X \times_C D & \xrightarrow{f'} & D \\
m' \downarrow & & \downarrow m \\
X & \xrightarrow{f} & C
\end{array}$$

Note that if m' is an isomorphism, then $g = f'(m')^{-1}$ will satisfy the condition $mg = f$. Now by restriction presheaf axioms and naturality of $\bar{\rho}$, we have $\theta \cdot_Q (m', mf') = \theta \cdot_Q (1, f)$. But $\theta \in (GQ)(C)$ implies

$$\overline{\theta \cdot_Q (m', mf')} = \overline{\theta} \circ (m', mf') = \overline{(m', mf')} = (m', m')$$

and

$$\overline{\theta \cdot_Q (1, f)} = \overline{\theta} \circ (1, f) = \overline{(1, f)} = (1, 1).$$

Therefore, m' must be an isomorphism, which means m and δ satisfy (4.3). Hence, G is also an \mathcal{M} -functor and $\text{PSh}_{\mathcal{M}}(\mathbf{C})$ and $\mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ are equivalent. \square

We now use the above theorem to prove the following result.

Proposition 33. *Let \mathbf{C} be an \mathcal{M} -category. Then there exists an equivalence of restriction categories $L: \text{Par}(\text{PSh}_{\mathcal{M}}(\mathbf{C})) \rightarrow \text{PSh}_r(\text{Par}(\mathbf{C}))$ satisfying the relation $\mathbf{y}_r = L \circ \text{Par}(\mathbf{y})$.*

Proof. Since Par and $\mathcal{M}\text{Total}$ are 2-equivalences, the following is an isomorphism of categories:

$$\mathcal{MCat}(\text{PSh}_{\mathcal{M}}(\mathbf{C}), \mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))) \cong \mathbf{rCat}(\text{Par}(\text{PSh}_{\mathcal{M}}(\mathbf{C})), \text{PSh}_r(\text{Par}(\mathbf{C}))).$$

We know from Theorem 32 that $F: \text{PSh}_{\mathcal{M}}(\mathbf{C}) \rightarrow \mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ is an equivalence. So define $L = \tilde{F}$, the transpose of F . Explicitly, $\tilde{F} = \Phi_{\text{PSh}_r(\text{Par}(\mathbf{C}))}^{-1} \circ \text{Par}(F)$, where $\Phi_{\text{PSh}_r(\text{Par}(\mathbf{C}))}$ is the unit of the Par and $\mathcal{M}\text{Total}$ 2-equivalence.

Now define $\tilde{\mathbf{y}}_r: \mathbf{C} \rightarrow \mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C})))$ as the transpose of $\mathbf{y}_r: \text{Par}(\mathbf{C}) \rightarrow \text{PSh}_r(\text{Par}(\mathbf{C}))$. Explicitly, $\tilde{\mathbf{y}}_r$ is the unique map making the following diagram commute:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\tilde{\mathbf{y}}_r} & \mathcal{M}\text{Total}(\text{PSh}_r(\text{Par}(\mathbf{C}))) \\ \downarrow & & \downarrow \\ \text{Par}(\mathbf{C}) & \xrightarrow{\mathbf{y}_r} & \text{PSh}_r(\text{Par}(\mathbf{C})) \end{array}$$

Since $\tilde{\mathbf{y}}_r = F\mathbf{y}$ will imply $\mathbf{y}_r = L \circ \text{Par}(\mathbf{y})$, we prove the former. So let $A \in \text{Par}(\mathbf{C})$. Then $\tilde{\mathbf{y}}_r(A) = \text{Par}(\mathbf{C})(-, A)$ by definition. On the other hand, $(F\mathbf{y})(A)$ defined on objects $B \in \text{Par}(\mathbf{C})$ is the following set:

$$(F\mathbf{y}A)(B) = \{(m, f) \mid m: Y \rightarrow B \in \mathcal{M}_{\mathbf{C}}, f \in \mathbf{C}(Y, A)\}.$$

In other words, elements of $(F\mathbf{y}A)(B)$ are spans

$$B \xleftarrow{m} Y \xrightarrow{f} A$$

Clearly $(F\mathbf{y}A)(B) = \text{Par}(\mathbf{C})(B, A) = (\tilde{\mathbf{y}}_r A)(B)$. Likewise, if $(n, g): C \rightarrow B$ is a morphism in $\text{Par}(\mathbf{C})$, then $(F\mathbf{y}A)(n, g) = (-) \circ (n, g) = (\tilde{\mathbf{y}}_r A)(n, g)$, and so $\tilde{\mathbf{y}}_r(A) = (F\mathbf{y})(A)$.

Now let $h: B \rightarrow C$ be a morphism in \mathbf{C} . Then $(F\mathbf{y})(h): \text{Par}(\mathbf{C})(-, B) \rightarrow \text{Par}(\mathbf{C})(-, C)$ has components given by

$$(F\mathbf{y}h)_D(n, g) = (n, (\mathbf{y}h)_{\text{dom } n}(g)) = (n, hg) = (1, h) \circ (n, g)$$

for all $D \in \text{Par}(\mathbf{C})$ and $(n, g) \in \text{Par}(\mathbf{C})(D, C)$. But $\tilde{\mathbf{y}}_r(h) = \mathbf{y}_r(1, h)$ also has components given by $(\mathbf{y}_r(1, h))_D = (1, h) \circ (-)$ at $D \in \text{Par}(\mathbf{C})$. Therefore, $(F\mathbf{y})(h) = \tilde{\mathbf{y}}_r(h)$ and so $F\mathbf{y} = \tilde{\mathbf{y}}_r$. Hence, $\mathbf{y}_r = L \circ \text{Par}(\mathbf{y})$. \square

We now prove the main result of this section.

Theorem 34. *Let \mathbf{X} be a restriction category. Then*

$$\mathbf{PSh}_r(\mathbf{X}) \simeq \mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))))$$

and the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} & \mathbf{X} & \\ \swarrow y_r & \cong & \searrow (3.6) \\ \mathbf{PSh}_r(\mathbf{X}) & \xleftarrow{\simeq} & \mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X})))) \end{array}$$

where (3.6) is the Cockett and Lack embedding.

Proof. Consider the following diagram, where $\mathbf{C} = \mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))$ and the top composite is (3.6):

$$\begin{array}{ccccc} \mathbf{X} & \xrightarrow{\Phi_{\mathbf{K}_r(\mathbf{X})} \circ J} & \mathbf{Par}(\mathbf{C}) & \xrightarrow{\mathbf{Par}(y)} & \mathbf{Par}(\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})) \\ y_r \downarrow & & \downarrow y_r & & \downarrow L \\ \mathbf{PSh}_r(\mathbf{X}) & \xleftarrow{(-) \circ (\Phi_{\mathbf{K}_r(\mathbf{X})} \circ J)^{\text{op}}} & \mathbf{PSh}_r(\mathbf{Par}(\mathbf{C})) & \xlongequal{\quad} & \mathbf{PSh}_r(\mathbf{Par}(\mathbf{C})) \end{array}$$

By Proposition 33, the right square commutes up to isomorphism. However, the left square also commutes up to isomorphism as $\Phi_{\mathbf{K}_r(\mathbf{X})} \circ J$ is fully faithful. Hence the result follows. \square

Corollary 35. *For any small restriction category \mathbf{X} , the embedding $y_r: \mathbf{X} \rightarrow \mathbf{PSh}_r(\mathbf{X})$ exhibits $\mathbf{PSh}_r(\mathbf{X})$ as the free restriction cocompletion of \mathbf{X} .*

5. FREE COCOMPLETION OF LOCALLY SMALL RESTRICTION CATEGORIES

So far in our discussions, we have considered the free cocompletion of a small \mathcal{M} -category \mathbf{C} and of a small restriction category \mathbf{X} , given by $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ and $\mathbf{PSh}_r(\mathbf{X})$ respectively. We now turn our attention to when our categories may not necessarily be small, but locally small. In the case where \mathbf{C} is an ordinary locally small category, we understand the category of small presheaves on \mathbf{C} , denoted by $\mathcal{P}(\mathbf{C})$, to be its free cocompletion [6]. Recall that a presheaf on \mathbf{C} is called small if it can be written as a small colimit of representables [6]. We would like to first give a notion of free cocompletion of locally small \mathcal{M} -categories, and then give an analogue in the restriction setting. To begin, we define what we mean by a locally small \mathcal{M} -category.

Definition 36 (Locally small \mathcal{M} -category). An \mathcal{M} -category $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ is called *locally small* if \mathbf{C} is locally small and \mathcal{M} -well-powered. That is, for any object in \mathbf{C} , the \mathcal{M} -subobjects of C form a small partially ordered set.

Remark 37. Note that this definition is exactly what is required for $\mathbf{Par}(\mathbf{C})$ to be a locally small restriction category when \mathbf{C} is a locally small \mathcal{M} -category, as noted by Robinson and Rosolini [11, p. 99].

Now we know when \mathbf{C} is an ordinary locally small category, $\mathcal{P}(\mathbf{C})$ is its free cocompletion. We also know that for any small \mathcal{M} -category $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$, its free cocompletion is given by $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C}) = (\mathbf{PSh}(\mathbf{C}), \mathcal{M}_{\mathbf{PSh}(\mathbf{C})})$. This suggests that for any locally small \mathcal{M} -category \mathbf{C} , we take $\mathcal{P}(\mathbf{C})$ as our base category and take its corresponding system of monics to be $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$, where $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ is defined in exactly the same way as for $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$. Call this pair $(\mathcal{P}(\mathbf{C}), \mathcal{M}_{\mathcal{P}(\mathbf{C})}) = \mathcal{P}_{\mathcal{M}}(\mathbf{C})$. However, it is not immediately obvious that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is an \mathcal{M} -category, since $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ may not be a stable system of monics. We therefore begin by showing that $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ is stable.

Lemma 38. *Let \mathbf{C} be a locally small \mathcal{M} -category, and let $\mu: P \Rightarrow Q$ be a map in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$. If $\gamma: Q' \Rightarrow Q$ is a map in $\mathbf{PSh}(\mathbf{C})$ with Q' a small presheaf, then the pullback of μ along γ is in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$.*

$$\begin{array}{ccc} P' & \longrightarrow & P \\ \mu' \downarrow & & \downarrow \mu \\ Q' & \xrightarrow{\gamma} & Q \end{array}$$

Proof. Certainly μ' exists and is in $\mathcal{M}_{\mathbf{PSh}(\mathbf{C})}$ by the fact $\mathbf{PSh}_{\mathcal{M}}(\mathbf{C})$ is an \mathcal{M} -category. So all we need to show is that P' is a small presheaf. Since Q' is small, we may rewrite $Q' \cong \text{colim } \mathbf{y}D$ for some functor $D: \mathbf{I} \rightarrow \mathbf{C}$ with \mathbf{I} small, and denote the colimiting coprojections as $q_I: \mathbf{y}D_I \rightarrow Q'$. Now μ is an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -map, which means that for each $I \in \mathbf{I}$ and composite $\gamma \circ q_I$, there exists an $m_I: C_I \rightarrow D_I$ making the outer square a pullback.

$$\begin{array}{ccccc} \mathbf{y}C_I & \xrightarrow{p_I} & P' & \longrightarrow & P \\ \mathbf{y}m_I \downarrow & & \mu' \downarrow & & \downarrow \mu \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q' & \xrightarrow{\gamma} & Q \end{array}$$

By the same argument as in the proof of Theorem 19, it follows that there is a functor $C: \mathbf{I} \rightarrow \mathbf{C}$ which on objects, takes I to C_I , and that there is a unique map $p_I: \mathbf{y}C_I \rightarrow P'$ making the left square a pullback for every $I \in \mathbf{I}$. However, because colimits are stable under pullback in $\mathbf{PSh}(\mathbf{C})$, this means $(p_I: \mathbf{y}C_I \rightarrow P')_{I \in \mathbf{I}}$ is colimiting, which ensures that P' is a small presheaf. \square

Remark 39. Note that the previous result implies that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -maps, and that these are computed pointwise.

Having now shown that $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ is a stable system of monics, and hence $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is an \mathcal{M} -category, we claim that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is indeed the free cocompletion of \mathbf{C} . To do so however, will first require showing that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is both locally small and cocomplete.

Lemma 40. *If \mathbf{C} is a locally small \mathcal{M} -category, then $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is locally small.*

Proof. It is well-known that $\mathcal{P}(\mathbf{C})$ is a locally small category [6], so all that remains is to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is \mathcal{M} -well-powered. So let Q be a small presheaf,

and rewrite $Q \cong \operatorname{colim} \mathbf{y}D$, where $D: \mathbf{I} \rightarrow \mathbf{C}$ is a functor with \mathbf{I} small. Again denote the colimiting coprojections by $(q_I: \mathbf{y}D_I \rightarrow Q)_{I \in \mathbf{I}}$.

As before, if $\mu: P \Rightarrow Q$ is an \mathcal{M} -subobject of Q , then there is a functor $C: \mathbf{I} \rightarrow \mathbf{C}$, which on objects, takes $I \rightarrow C_I$, and unique maps $(p_I: \mathbf{y}C_I \rightarrow P)_{I \in \mathbf{I}}$ making the following squares pullbacks for each $I \in \mathbf{I}$:

$$\begin{array}{ccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \\ \mathbf{y}m_I \downarrow & & \downarrow \mu \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q \end{array}$$

Note that $P \cong \operatorname{colim} \mathbf{y}C$ as colimits are stable under pullback in $\mathcal{P}(\mathbf{C})$. There is also a natural transformation $\alpha: C \Rightarrow D$, given componentwise on I by $m_I \in \mathcal{M}_{\mathbf{C}}$ and whose naturality squares are pullbacks for every $I \in \mathbf{I}$. In fact, these functors from \mathbf{I} to \mathbf{C} form an \mathcal{M} -category $(\mathbf{C}^{\mathbf{I}}, \mathcal{M}_{\mathbf{C}^{\mathbf{I}}})$ whose $\mathcal{M}_{\mathbf{C}^{\mathbf{I}}}$ -maps are just the natural transformations whose components are $\mathcal{M}_{\mathbf{C}}$ -maps. Note that by observation, this \mathcal{M} -category $(\mathbf{C}^{\mathbf{I}}, \mathcal{M}_{\mathbf{C}^{\mathbf{I}}})$ is locally small.

It is not then difficult to see there is a function $f: \mathbf{Sub}_{\mathcal{M}_{\mathcal{P}(\mathbf{C})}}(Q) \rightarrow \mathbf{Sub}_{\mathcal{M}_{\mathbf{C}^{\mathbf{I}}}}(D)$ taking the \mathcal{M} -subobjects of Q to the \mathcal{M} -subobjects of D . So to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is \mathcal{M} -well-powered, it is enough to show that f is injective. Let $\mu: P \Rightarrow Q$ and $\mu': P' \Rightarrow Q$ be two \mathcal{M} -subobjects of Q which are mapped to the same \mathcal{M} -subobject of D . That is, there is an isomorphism from C to C' making the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\cong} & C' \\ & \searrow \alpha & \swarrow \alpha' \\ & D & \end{array}$$

But because $P \cong \operatorname{colim} \mathbf{y}C \cong \operatorname{colim} \mathbf{y}C' \cong P'$, this induces an isomorphism between P and P' making the following diagram commute:

$$\begin{array}{ccccc} \mathbf{y}C_I & \xrightarrow{p_I} & P & & \\ & \searrow \cong & \downarrow \mu & \swarrow \cong & \\ & & \mathbf{y}C'_I & \xrightarrow{p'_I} & P' \\ \mathbf{y}\alpha_I \downarrow & \swarrow & \downarrow \mu & \swarrow \mu' & \\ \mathbf{y}D_I & \xrightarrow{q_I} & Q & & \end{array}$$

In other words, μ and μ' are the same \mathcal{M} -subobject of Q , and so the function f is injective. Hence, if \mathbf{C} is a locally small \mathcal{M} -category, then so is $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$. \square

Next, to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is cocomplete, we exploit Proposition 11 and the following two lemmas.

Lemma 41. *Let \mathbf{C} be a locally small \mathcal{M} -category. If $\{\mu_i: P_i \rightarrow Q_i\}_{i \in I}$ is a family of maps in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ indexed by a small set I , then so is their coproduct $\sum_{i \in I} \mu_i$.*

Proof. Let $\{\mu_i: P_i \rightarrow Q_i\}_{i \in I}$ be a family of maps in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$, with I some small set. To show that $\sum_{i \in I} \mu_i$ is also in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$, we need to show that for any

$h: \mathbf{y}D \rightarrow \sum_{i \in I} Q_i$, there is a map $m: C \rightarrow D$ in $\mathcal{M}_{\mathbf{C}}$ making the following diagram a pullback:

$$\begin{array}{ccc} \mathbf{y}C & \longrightarrow & \sum_{i \in I} P_i \\ \mathbf{y}m \downarrow & & \downarrow \sum_{i \in I} \mu_i \\ \mathbf{y}D & \xrightarrow{h} & \sum_{i \in I} Q_i \end{array}$$

By Yoneda, there is a bijection

$$\mathcal{P}(\mathbf{C}) \left(\mathbf{y}D, \sum_{i \in I} Q_i \right) \cong \left(\sum_{i \in I} Q_i \right) (D) \cong \sum_{i \in I} Q_i D,$$

meaning that h corresponds uniquely with some element $\tilde{h} \in \sum_{i \in I} Q_i D$. For each $i \in I$, the naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q_i) \cong Q_i D$ implies that $h: \mathbf{y}D \rightarrow \sum_{i \in I} Q_i$ factors through exactly one of the coproduct injections $\iota_{Q_j}: Q_j \rightarrow \sum_{i \in I} Q_i$. By extensivity of the presheaf category $\mathbf{PSh}(\mathbf{C})$, the pullback of $\sum_{i \in I} \mu_i$ along ι_{Q_i} must be μ_i . However, as μ_j is an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -map, there exists an $m: C \rightarrow D$ in $\mathcal{M}_{\mathbf{C}}$ making the left square of the following diagram commute:

$$\begin{array}{ccccc} \mathbf{y}C & \longrightarrow & P_j & \xrightarrow{\iota_{P_j}} & \sum_{i \in I} P_i \\ \mathbf{y}m \downarrow & & \mu_j \downarrow & & \downarrow \sum_{i \in I} \mu_i \\ \mathbf{y}D & \xrightarrow{h'} & Q_j & \xrightarrow{\iota_{Q_j}} & \sum_{i \in I} Q_i \\ & \searrow h & & & \end{array}$$

Therefore, as both squares are pullbacks, $\mathbf{y}m$ is a pullback of $\sum_{i \in I} \mu_i$ along h , which means $\sum_{i \in I} \mu_i \in \mathcal{M}_{\mathcal{P}(\mathbf{C})}$. \square

Lemma 42. *Let \mathbf{C} be a locally small \mathcal{M} -category, and suppose m is a map in $\mathcal{P}(\mathbf{C})$. If the pullback of m along some epimorphism is an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -map, then m must also be in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$.*

Proof. Let $m: P \Rightarrow Q$ be a map in $\mathcal{P}(\mathbf{C})$, and suppose $m': P' \Rightarrow Q'$ is a pullback of m along some epimorphism $f: Q' \Rightarrow Q$. To show that m is an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$, let $g: \mathbf{y}D \Rightarrow Q$ be any map in $\mathcal{P}(\mathbf{C})$. Again by Yoneda, there is a bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$, giving a corresponding element $\tilde{g} \in QD$. Since f is an epimorphism in $\mathcal{P}(\mathbf{C})$, its component at D , $f_D: Q'D \rightarrow QD$, must also be an epimorphism, which means there exists some element $\tilde{f}' \in Q'D$ such that $f_D(\tilde{f}') = \tilde{g}$. The naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$ then implies there is a map $f': \mathbf{y}D \Rightarrow Q'$ such that $g = f f'$. Now using the fact m' is an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -map, there exists a map $n \in \mathcal{M}_{\mathbf{C}}$ such that $\mathbf{y}n$ is the pullback of m' along f' .

$$\begin{array}{ccccc}
\mathbf{y}C & \longrightarrow & P' & \longrightarrow & P \\
\mathbf{y}n \downarrow & & m' \downarrow & & \downarrow m \\
\mathbf{y}D & \xrightarrow{f'} & Q' & \xrightarrow{f} & Q \\
& & \searrow g & &
\end{array}$$

Then as both squares are pullbacks, $\mathbf{y}n$ must be the pullback of m along $g = ff'$, making m an $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -map. \square

We now prove that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is indeed cocomplete as an \mathcal{M} -category.

Lemma 43. *Let $(\mathbf{C}, \mathcal{M}_{\mathbf{C}})$ be a locally small \mathcal{M} -category. Then $(\mathcal{P}(\mathbf{C}), \mathcal{M}_{\mathcal{P}(\mathbf{C})})$ is a cocomplete \mathcal{M} -category.*

Proof. We begin by noting that the category of small presheaves on \mathbf{C} , $\mathcal{P}(\mathbf{C})$, is cocomplete. Therefore, it remains to show that the inclusion $\mathcal{P}(\mathbf{C}) \hookrightarrow \text{Par}(\mathcal{P}(\mathbf{C}), \mathcal{M}_{\mathcal{P}(\mathbf{C})})$ is cocontinuous. However, by Proposition 11, it is enough to show that the following conditions hold:

- (a) If $\{m_i: P_i \Rightarrow Q_i\}_{i \in I}$ is a family of small I -indexed set of maps in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$, then $\sum_{i \in I} m_i$ is also in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ and the following squares are pullbacks for each $i \in I$:

$$\begin{array}{ccc}
P_i & \xrightarrow{\iota_{P_i}} & \sum_{i \in I} P_i \\
m_i \downarrow & & \downarrow \sum_{i \in I} m_i \\
Q_i & \xrightarrow{\iota_{Q_i}} & \sum_{i \in I} Q_i
\end{array}$$

- (b) Given the following diagram,

$$\begin{array}{ccccc}
P' & \xrightarrow{f'} & P & \xrightarrow{c'} & G \\
m' \downarrow & & \downarrow m & & \downarrow n \\
Q' & \xrightarrow{f} & Q & \xrightarrow{c} & H
\end{array}$$

if $m \in \mathcal{M}_{\mathcal{P}(\mathbf{C})}$ and the left two squares are pullbacks, and c, c' are the coequalisers of f, g and f', g' respectively, then the unique map n making the right square commute is in $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ and the right square is also a pullback.

- (c) Colimits in $\mathcal{P}(\mathbf{C})$ are stable under pullback along $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -maps.

To see that (c) holds, recall that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{M}_{\mathcal{P}(\mathbf{C})}$ -maps, and that these are calculated pointwise as in **Set** (Remark 39). The result then follows from the fact that colimits in $\mathcal{P}(\mathbf{C})$ are also calculated pointwise together with the fact colimits are stable under pullback in **Set**.

For (b), it will be enough to show that the square on the right in (b) is a pullback (by Lemma 42). Now the right square is a pullback in $\mathcal{P}(\mathbf{C})$ if and only if componentwise for every $A \in \mathbf{C}$, it is a pullback in **Set**. So consider the diagram in (b) componentwise at $A \in \mathbf{C}$:

$$\begin{array}{ccccc}
P'A & \xrightarrow{f'_A} & PA & \xrightarrow{c'_A} & GA \\
m'_A \downarrow & g'_A & \downarrow m_A & & \downarrow n_A \\
Q'A & \xrightarrow[f_A]{g_A} & QA & \xrightarrow{c_A} & HA
\end{array}$$

The two left squares remain pullbacks in **Set**, and c_A, c'_A remain coequalisers of f_A, g_A and f'_A, g'_A respectively since colimits in $\mathcal{P}(\mathbf{C})$ are calculated pointwise. Observe also that m_A is a monomorphism as maps between small presheaves in $\mathcal{P}(\mathbf{C})$ are monic if and only if they are componentwise monic for every $A \in \mathbf{C}$ (by a Yoneda argument). Now we know that the \mathcal{M} -category $(\mathbf{Set}, \mathcal{M}_{\mathbf{Set}})$ (where $\mathcal{M}_{\mathbf{Set}}$ are all the injective functions) is a cocomplete \mathcal{M} -category (Example 7), and since m_A is monic, the square on the right must be a pullback in **Set**. Therefore, as pullbacks in $\mathcal{P}(\mathbf{C})$ are calculated pointwise, the square on the right of (b) must also be a pullback.

For (a), we know that $\sum_{i \in I} m_i \in M_{\mathcal{P}(\mathbf{C})}$ from Lemma 41. Then, as $(\mathbf{Set}, \mathcal{M}_{\mathbf{Set}})$ is cocomplete and both pullbacks and colimits in $\mathcal{P}(\mathbf{C})$ are computed pointwise as in **Set**, the result follows by an analogous argument to (b).

Therefore, $(\mathcal{P}(\mathbf{C}), \mathcal{M}_{\mathcal{P}(\mathbf{C})})$ is a cocomplete \mathcal{M} -category. \square

Theorem 44. (*Free cocompletion of locally small \mathcal{M} -categories*) Let \mathbf{C} be a locally small \mathcal{M} -category, and let \mathbf{D} be a locally small, cocomplete \mathcal{M} -category. Then the following is an equivalence of categories:

$$(-) \circ \mathbf{y}: \mathcal{M}\mathbf{Cocomp}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \rightarrow \mathcal{M}\mathbf{CAT}(\mathbf{C}, \mathbf{D})$$

where $\mathcal{M}\mathbf{CAT}$ is the 2-category of locally small \mathcal{M} -categories.

Proof. The proof follows exactly the same arguments presented in the proof of Theorem 19. \square

Corollary 45. (*Free cocompletion of locally small restriction categories*) For any locally small restriction category \mathbf{X} and locally small, cocomplete restriction category \mathcal{E} , the following is an equivalence of categories:

$$(-) \circ (3.6): \mathbf{rCocomp}(\mathbf{Par}(\mathcal{P}_{\mathcal{M}}(\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X})))), \mathcal{E}) \rightarrow \mathbf{rCAT}(\mathbf{X}, \mathcal{E})$$

where (3.6) is the Cockett and Lack embedding and \mathbf{rCAT} is the 2-category of locally small restriction categories.

Given that a small presheaf on an ordinary category is one that can be written as a colimit of small representables, it is natural to ask whether there is a similar notion of small restriction presheaf. So let \mathbf{X} be a locally small restriction category, and denoting the \mathcal{M} -category $\mathcal{M}\mathbf{Total}(\mathbf{K}_r(\mathbf{X}))$ by \mathbf{C} , the previous corollary says that $\mathbf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}))$ is the free cocompletion of \mathbf{X} . Since $\mathcal{P}(\mathbf{C})$ is a full replete subcategory of $\mathbf{PSh}(\mathbf{C})$, and $\mathbf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C})) \simeq \mathbf{PSh}_r(\mathbf{X})$, there exists a full subcategory of $\mathbf{PSh}_r(\mathbf{X})$ which is equivalent to $\mathbf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}))$:

$$\begin{array}{ccc}
\mathcal{P}_r(\mathbf{X}) & \xrightarrow{\simeq} & \text{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C})) \\
\downarrow & & \downarrow \\
\text{PSh}_r(\mathbf{X}) & \xrightarrow{\simeq} & \text{Par}(\text{PSh}_{\mathcal{M}}(\mathbf{C}))
\end{array}$$

where the above square is a pullback and the bottom map is the equivalence from Theorem 34.

To see what objects should be in $\mathcal{P}_r(\mathbf{X})$, it is enough to apply Total to the above diagram, giving the following pullback:

$$\begin{array}{ccc}
\text{Total}(\mathcal{P}_r(\mathbf{X})) & \longrightarrow & \mathcal{P}(\text{Total}(\mathbf{K}_r(\mathbf{X}))) \\
\downarrow & & \downarrow \\
\text{Total}(\text{PSh}_r(\mathbf{X})) & \xrightarrow{G} & \text{PSh}(\text{Total}(\mathbf{K}_r(\mathbf{X})))
\end{array}$$

where G is an equivalence. Since the above diagram is a pullback, an object P will be in $\text{Total}(\mathcal{P}_r(\mathbf{X}))$ (and hence in $\mathcal{P}_r(\mathbf{X})$) if GP is an object in $\mathcal{P}(\text{Total}(\mathbf{K}_r(\mathbf{X})))$; that is, $GP \cong \text{colim } \mathbf{y}C_I$, where $C: \mathbf{I} \rightarrow \text{Total}(\mathbf{K}_r(\mathbf{X}))$ is a functor with \mathbf{I} small. If we define H to be a pseudo-inverse for G , then an object will be in $\mathcal{P}_r(\mathbf{X})$ if it is of the form $P \cong \text{colim } H\mathbf{y}C_I$, for some small \mathbf{I} and functor $C: \mathbf{I} \rightarrow \text{Total}(\mathbf{K}_r(\mathbf{X}))$. We call these P the *small restriction presheaves*.

We also give an explicit description of a small restriction presheaf as follows. Since GP is an object in $\mathcal{P}(\text{Total}(\mathbf{K}_r(\mathbf{X})))$, it will be the colimit of a small diagram whose vertices are of the form $\mathbf{y}(A, e)$, where (A, e) is an object in $\mathbf{K}_r(\mathbf{X})$. Now given $(A, e) \in \mathbf{K}_r(\mathbf{X})$, note the following splitting in $\text{PSh}_r(\mathbf{X})$:

$$\begin{array}{ccc}
& Q(A, e) & \\
\nearrow & & \searrow \\
\mathbf{y}_r A & \xrightarrow{\mathbf{y}_r e} & \mathbf{y}_r A
\end{array}$$

This gives a functor $Q: \mathbf{K}_r(\mathbf{X}) \rightarrow \text{PSh}_r(\mathbf{X})$. Then a restriction presheaf is called *small* if it is the colimit of some functor $D: \mathbf{I} \rightarrow \text{PSh}_r(\mathbf{X})$ (\mathbf{I} small), where each DI is of the form $Q(A, e)$ for some $(A, e) \in \mathbf{K}_r(\mathbf{X})$, and each $D(f: I \rightarrow J)$ is total. We denote by $\mathcal{P}_r(\mathbf{X})$ the restriction category whose objects are small restriction presheaves on \mathbf{X} . By construction, it is also the free cocompletion of \mathbf{X} . It is not difficult to check that when \mathbf{X} is a small restriction category, restriction presheaves on \mathbf{X} are small, and so $\mathcal{P}_r(\mathbf{X}) = \text{PSh}_r(\mathbf{X})$.

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